Integer Programming Models for the Target Visitation Problem

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ABSTRACT

The target visitation problem (TVP) is concerned with finding a route to visit a set of targets starting from and returning to some base. In addition to the distance traveled a tour is evaluated by taking also preferences into account which address the sequence in which the targets are visited. The problem thus is a combination of two well-known combinatorial optimization problems: the traveling salesman and the linear ordering problem. In this paper we present several possible IP-Models for this problem and compared them to their usability for branch-and-cut approaches.

1. INTRODUCTION

Let $D_{n+1} = (V_{n+1}, A_{n+1})$ be the complete digraph on $n+1$ nodes where we set $V_{n+1} = \{0, 1, \ldots, n\}$. Furthermore let two types of arc weights be defined: weights d_{ij} (*distances*) for every arc (i, j), $0 \le i, j \le n$, and weights p_{ij} (*preferences*) associated with every arc (i, j), $1 \le i, j \le n$. The *target visitation problem (TVP)* consists of finding a Hamiltonian tour starting at node 0 visiting all remaining nodes (called *targets*) exactly once in some order and returning to node 0. Every tour can be represented by a permutation π of $\{1, 2, \ldots, n\}$ where $\pi(i) = j$ if target j is visited as the i-th target. For convenience we also define $\pi(0) = 0$ and $\pi(n+1) = 0$.

So we are essentially looking for a traveling salesman tour, but for the TVP the profit of a tour depends on the two weights. Namely, the value of a tour is the sum of pairwise preferences between the targets corresponding to their visiting sequence minus the sum of distances traveled, i.e., it is calculated as

$$\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} p_{\pi(i)\pi(j)} - \sum_{i=0}^{n} d_{\pi(i)\pi(i+1)},$$

and the task is to find a tour of maximum value. So we have a multicriterial objective function.

The TVP was introduced in [3] and combines two classical combinatorial optimization problems: the *asymmetric traveling salesman problem (ATSP)* asking for a shortest Hamiltonian tour and the *linear ordering problem (LOP)* which is to find an acyclic tournament of maximum weight. (There is an obvious 1–1 correspondence between acyclic tournaments and linear orders of the node). Computational results of a genetic algorithm for problem instances with up to 16 targets have been reported in [1]. The original application of the TVP is the planning of routes for UAVs (unarmed aerial vehicles). But there is a wide field of applications, e.g., the delivery of relief supplies or any other routing problem where additional preferences should be considered (town cleaning, snow-plowing service, etc.).

Obviously, the TVP is NP-hard because it contains the traveling salesman problem ($p \equiv 0$) and the linear ordering problem ($d \equiv 0$) as special cases.

For convenience we transform the problem to a Hamiltonian path problem and also get rid of the special base node. This transformation is well-known for the ATSP [6] and can be adapted for the TVP as follows.

The key idea is to exploit the fact that each tour has to start at the base and return to it and that no preferences are to be taken into account for the base. In the *TVP-path model* we leave out this node and just search for a Hamiltonian path which visits all targets exactly once.

Following [6] we make the following modifications.

(i) Transform the distance matrix by setting $d'_{ij} = d_{ij} - d_{i0} - d_{0j}$, for all pairs i and j of nodes, $1 \le i, j \le n$ $i \ne j$.

(ii) Change the computation of the distance part of the

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objective function to

$$\sum_{i=1}^{n-1} d'_{\pi(i)\pi(i+1)} - \sum_{i=1}^{n} d_{i0} - \sum_{i=1}^{n} d_{0i}.$$

The preferences are not affected by this change. From now on we consider the TVP as finding a Hamiltonian path in the complete digraph $D_n = (V_n, A_n)$ with additional preference costs to be taken into account. The path is described by a permutation π of $\{1, \ldots, n\}$ where $\pi(k)$ is the node at position k.

In this paper we want to present four different IP-models and compare them to there usability for branch-and-cut approaches

2. AN INTEGER PROGRAMMING MODEL FOR THE TVP

Our first approach for obtaining an IP model of the TVP is to combine well-known IP formulations for the ATSP and the LOP. For this purpose we have to introduce the following two types of variables.

The sequence in which the targets are visited is represented by binary *ATSP variables* x_{ij} for $1 \leq i, j \leq n,\ i \neq j$, with the interpretation

$$x_{ij} := \begin{cases} 1 & \text{if } i = \pi(k) \text{ and } j = \pi(k+1) \\ & \text{for some } 1 \leq k \leq n-1, \\ 0 & \text{otherwise.} \end{cases}$$

The fact that some target i is visited before some target j is modeled with binary *LOP variables* w_{ij} for $1 \leq i, j \leq n,\ i \neq j$, with the definition

$$w_{ij} := \begin{cases} 1 & \text{if } i = \pi(k) \text{ and } j = \pi(l) \text{ for some } 1 \leq k < l \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

Using this variables we can formulate the following IP-model for the TVP, which we will denote by TVP-IP$_1$

$$\max \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} p_{ij} w_{ij} - \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} d_{ij} x_{ij} \tag{1}$$

$$\sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} x_{ij} = n-1, \tag{2}$$

$$\sum_{i \in S} \sum_{\substack{j \in S \\ j \neq i}} x_{ij} \leq |S| - 1,\ S \subseteq \{1, \ldots, n\}, 2 \leq |S| \leq n-1, \tag{3}$$

$$\sum_{\substack{i=1 \\ i \neq j}}^{n} x_{ij} \leq 1, \qquad 1 \leq j \leq n, \tag{4}$$

$$\sum_{\substack{j=1 \\ j \neq i}}^{n} x_{ij} \leq 1, \qquad 1 \leq i \leq n, \tag{5}$$

$$w_{ij} + w_{jk} + w_{ki} \leq 2,\ 1 \leq i, j, k \leq n,\ i < j, i < k\ j \neq k, \tag{6}$$

$$w_{ji} + w_{ij} = 1, \qquad 1 \leq i, j \leq n,\ i \neq j \tag{7}$$

$$x_{ij} - w_{ij} \leq 0, \qquad 1 \leq i, j \leq n,\ i \neq j \tag{8}$$

$$x_{ij}, w_{ij} \in \{0, 1\}, \quad 1 \leq i, j \leq n,\ i \neq j \tag{9}$$

Constraints (2)–(5) model the directed Hamiltonian paths where inequalities (3) are the *subtour elimination constraints*. Acyclic tournaments are modelled by the *3-dicycle inequalities* (6) and the *tournament equations* (7). Inequalities (8) connect the solutions of both problems. Together with the integrality conditions (9) this obviously constitutes a 0/1 model of the TVP.

As an interesting fact we note that the subtour elimination constraints are actually not needed. If (w, x) satisfies (2) and (4)–(9), but not all inequalities (3) then there is some subtour on $k \geq 2$ nodes. W.l.o.g. we can assume that the node set is $\{1, 2, \ldots, k\}$ and the subtour is given as $\{(1,2), (2,3), \ldots (k-1,k), (k,1)\}$. Hence $x_{12} = x_{23} = \ldots = x_{k-1,k} = x_{k1} = 1$, implying because of (8) that $w_{12} = w_{23} = \ldots = w_{k-1,k} = w_{k1} = 1$. This is a contradiction to the requirement that the w-variables represent an acyclic tournament.

So we can eliminate the exponentially many constraints (3) and obtain a TVP formulation with a polynomial number (cubic in n) of constraints.

Note that because of the tournament equations we can substitute an LOP variable w_{ij}, $j > i$, by $1 - w_{ji}$. Now the 3-dicycle inequalities are turned into $1 \geq w_{ij} + w_{jk} - w_{ik} \geq 0$ for all $1 \leq i < j < k \leq n$ and the part of the objective function for the LOP variables reads $\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} [(p_{ij} - p_{ji}) w_{ij} + p_{ji}]$.

3. THE EDGE-NODE-FORMULATION

The key idea of the next model is to combine the w and x variables of the first model to new three index variables which state the relation between a node n and an fixed edge

(i,j). More precisely we define:

$$w_{ij}^k := \begin{cases} 1 & \text{if } k=\pi(a),\ i=\pi(b) \text{ and } j=\pi(b+1) \\ & \text{for some } 1 \le a \le b \le n-1 \\ 0 & \text{otherwise.} \end{cases}$$

and analogously:

$$w_k^{ij} := \begin{cases} 1 & \text{if } i=\pi(a),\ j=\pi(a+1) \text{ and } k=\pi(b) \\ & \text{for some } 1 \le a < b \le n \\ 0 & \text{otherwise.} \end{cases}$$

A valid IP model based one these variables can be formulated as follows. We denote this model by TVP-IP$_2$.

$$max \sum_{i=1}^{n}\sum_{\substack{j=1\\ i\neq j}}^{n} p_{ij}(\sum_{\substack{m=1\\ m\neq j}}^{n} w_{mj}^{i}) + \sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}(w_{ij}^{\Omega} + w_{\Omega}^{ij}) \tag{10}$$

$$s.t.$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n}(w_{ij}^{\Omega} + w_{\Omega}^{ij}) = n-1, \tag{11}$$

$$\sum_{i=1}^{n}(w_{ij}^{\Omega} + w_{\Omega}^{ij}) \le 1, \quad j \in V \tag{12}$$

$$\sum_{j=1}^{n}(w_{ij}^{\Omega} + w_{\Omega}^{ij}) \le 1, \quad i \in V \tag{13}$$

$$\sum_{\substack{l=1\\ l\neq j}}^{n} w_{lj}^{i} + \sum_{\substack{l=1\\ l\neq k}}^{n} w_{lk}^{j} + \sum_{\substack{l=1\\ l\neq i}}^{n} w_{li}^{k} + (w_{ik}^{\Omega} + w_{\Omega}^{ik}) \le 2, \quad i,j,k \in V \tag{14}$$

$$w_{ij}^k, w_k^{ij} \in \{0,1\}, \ i,j,k \in V \tag{15}$$

Please note that that $\Omega \in V$ could be chosen arbitrarily for each summand in (10) -(13) but must be $\neq j$ and $\neq i$ in each case. It is the same in (14) but here Ω must not be equal i or k. For the sake of uniqueness we always choose the Ω in general as small as possible.

4. THREE DISTANCE MODEL

Another idea for constructing an IP-model for the TVP has been made by Prof. E. Fernandez from the UPC Barcelona. The key idea of this approach is the use of distance variables. In detail we define variables z_{ij}^t which describe the fact whether there exists a path of length t between i an j or not. More formally we state:

$$z_{ij}^t = \begin{cases} 1 & \text{if the solution contains a path with } t \text{ arcs} \\ & \text{from } i \text{ to } j, \\ 0 & \text{otherwise.} \end{cases}$$

Since we do not longer distinguish between distance and ordering variables we have to adjust the coefficients in the following way:

$$w_{ij}^t = \begin{cases} c_{ij} - d_{ij} & \text{if } t = 1, \\ c_{ij} & \text{otherwise.} \end{cases}$$

With this modification we are now able to formulated a TVP model with distance variables, which we are denote by TVP-IP$_3$.

$$\max \sum_{i\in N}\sum_{j\in N\setminus\{i\}}\sum_{t\in N\setminus\{n\}} \overline{w}_{ij}^t z_{ij}^t \tag{16}$$

$$\sum_{\substack{i=1\\ i\neq j}}^{n} z_{ij}^1 \le 1 \qquad j \in N, \tag{17}$$

$$\sum_{\substack{j=1\\ j\neq i}}^{n} z_{ij}^1 \le 1 \qquad i \in N, \tag{18}$$

$$\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n} z_{ij}^k = n-k \qquad k \in V \tag{19}$$

$$z_{ij}^{t_1} + z_{jk}^{t_2} \le z_{ik}^{t_1+t_2} + 1, \quad i,j,k \in N, t_1,t_2 \in N \setminus \{n\}, i \neq j \neq k, t_1+t_2 < n, \tag{20}$$

$$\sum_{t=1}^{n-1} z_{ij}^t + z_{ji}^t = 1 \qquad i,j \in N, i \neq j, \tag{21}$$

$$z_{ij}^t \in \{0,1\} \qquad i,j \in N, i \neq j, t \in N \setminus \{n\}. \tag{22}$$

As one can see, we only have just one type of variables in this model but nevertheless the $z_{i,j}^1$ variables play still a special role in the objective function.

4.1 Extended Formulation of the Basic Model

Our last model is an extended formulation of our first model

The use of extended formulations is a common technique for to strengthen the LP formulation of a combinatorial optimization problem. The key idea of this approach is to add new variables and constraints to a given IP formulation in order to reduce the gap between the solution of the LP relaxation and the optimal integral solution.

In the case of TVP we can obtain an extended formulation by adding three-indexed variables, which are a generalization of the linear ordering variables of the first model. In detail these new variables w_{ijk} are defined as follows:

$$w_{ijk} := \begin{cases} 1 & \text{if } i=\pi(a),\ j=\pi(b) \text{ and } k=\pi(c) \\ & \text{for some } 1 \le a < b < c \le n \\ 0 & \text{otherwise.} \end{cases}$$

So as one ca see this new type of variables is a straight forward extension of the w_{ij}-variables. In the objective function we assign zero coefficients to the new variables. In order to extend our standard model we also need to introduce two new classes of constraints to make sure that the solution of the new variables matches with the old x_{ij} and w_{ij} variables. In detail the extended formulation (denoted by TVP-IP$_4$) looks as follows:

$$\max \sum_{i=1}^{n}\sum_{\substack{j=1\\j\neq i}}^{n} p_{ij}w_{ij} - \sum_{i=1}^{n}\sum_{\substack{j=1\\j\neq i}}^{n} d_{ij}x_{ij} \tag{23}$$

$$s.t.$$

$$\sum_{i=1}^{n}\sum_{\substack{j=1\\j\neq i}}^{n} x_{ij} = n-1, \tag{24}$$

$$\sum_{i=1}^{n} x_{ij} \leq 1,\ 1 \leq j \leq n, \tag{25}$$

$$\sum_{j=1}^{n} x_{ij} \leq 1,\ 1 \leq i \leq n, \tag{26}$$

$$w_{ij} + w_{jk} + w_{ki} \leq 2, 1 \leq i,j,k \leq n,\ i<j, i<k\ j \neq k, \tag{27}$$

$$w_{ij} + w_{jik} + w_{jki} + w_{kji} = 1,\ 1 \leq i,j,k \leq n, i<j \tag{28}$$

$$x_{ij} - w_{ijk} - w_{kij} \leq 0,\ 1 \leq i,j,k \leq n, i<j \tag{29}$$

$$x_{ij} - w_{ij} \leq 0,\ 1 \leq i,j \leq n, \tag{30}$$

$$w_{ij}, x_{ij} \in \{0,1\},\ 1 \leq i,j \leq n, \tag{31}$$

$$w_{ijk} \in \{0,1\},\ 1 \leq i,j,k \leq n. \tag{32}$$

5. COMPUTATIONAL RESULTS

One very important criterion for the usability of a given IP model in a cutting plane or branch-and-cut approach is the gap between the LP relaxation and the optimal solution of the IP. For this purpose we tested our models on several instances (taken from [4]), which were randomly created (coefficient matrices have entries between 0 and 10, uniformly distributed). The results can be seen in Table 1.

Instance	Opt.	TVP-IP$_1$	TVP-IP$_2$	TVP-IP$_3$	TVP-IP$_4$
HP13	410	430.8	426.0	478.5	416.8
HP16	604	631.0	622.2	702.2	611.7
HP19	894	921.4	913.9	1038.0	903.1
HP21	1046	1077.6	1070.0	1246.2	1057.1
HP23	1302	1333.0	1321.7	1512.8	1313.8

Table 1: Comparison of root bounds of the different TVP models

As one can see the gap of the model TVP-IP$_1$ is larger than for the TVP-IP$_2$ model. The extended formulation TVP-IP$_4$ has as expected a much better gap. The model with distance variables has in the average a really large gap. Considering the fact that this model also contains a really large number of constraints this model seems to be not so practical unless we find a way to strengthen the LP-relaxation and develop a fast separation algorithm for constraints number (20). We like to point out that these results remain also true if we are considering real world instances.

Another interesting point is the comparison of the polyhedral description for small polytopes of the different models. So we compute such small polytopes for the remaining three polytopes with PORTA[2] and classify the facets due to symmetry with the help of HUHFA[5]. The results could been seen in Table 1.

Polytope	Dim.	# Facets	# Facet-Classes
TVP-IP$_1$	4	1280	48
TVP-IP$_2$	4	512	24
TVP-IP$_3$	4	144	7

Table 2: Key facts of the different TVP-Polytopes

For the the model TVP-IP$_1$ we were able to prove that $w_{ij} + w_{jk} + w_{ki} + x_{ik} \leq 2$ is a facet for all n. So we can strengthen this model by replacing the normal three cycles by this facet. The resulting root bounds are now nearly as good as for the TVP-IP$_2$ model.

HP13	HP16	HP19	HP21	HP23
427.4	621.6	914.4	1069.9	1322.8

Table 3: Rootsbound for TVP-IP$_1$ with the use of the extended three cycle

As thing last we implemented a branch-and-cut for the three models. We made the following observations: The TVP-IP$_1$ model behaves much better than the TVP-IP$_2$ model which is no surprise since the first one has a quadratic number of variables. It is also much faster compared to the last model. Unfortunately this remains also true if you add pricing to the algorithm. So nevertheless the last model has much better root bounds we are currently not able to use this fact efficiently.

So if one sums up all this information, the first model still seems the one which is most suitable for practical computations. Nevertheless the other two model have an interesting structure and are worth further theoretical studies.

6. REFERENCES

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