UNIVERSITY OF HEIDELBERG
Institute of Computer Science

Discrete and Combinatorial Optimization

Linear Ordering Problem

Motivation

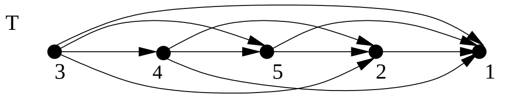
Suppose m persons have assessed n objects $O_1, \dots, O_n$ by pairwise comparisons and that based on these judgments a ranking of the objects has to be found. One possibility is to determine a linear ordering of the objects such that the number of individual pairwise assignments that are not in accordance with this ordering is minimized. This problem is known as the *linear ordering problem* and belongs to the class of *NP*-hard combinatorial optimization problems.

Applications

- **Economics:** triangulation problem for input-output tables
- **Social science:** aggregation of individual preferences
- **Politics:** evaluation of corruption perception index
- **Industry:** determination of the best schedule on a single machine with restrictions of the tasks among each other
- **Sports:** optimum triangulation of summary tables

Mathematical formulation

- Let $D_n = (V_n, A_n)$ be the complete digraph on n nodes and $m = n(n-1)$ arcs. A *tournament* T in A_n is a subset of arcs containing, for every pair of nodes i, j, either the arc (i,j) or the arc (j,i) but not both. An *acyclic* tournament is a tournament without directed cycles. It is easy to see, that such an acyclic tournament corresponds to a linear ordering and vice versa.



The permutation π corresponding to the acyclic tournament T is $\pi = (3\ 4\ 5\ 2\ 1)$.

- The linear ordering polytope P^n_{LO} is defined as the convex hull of the incidence vectors of the acyclic tournaments in D_n, i.e.,

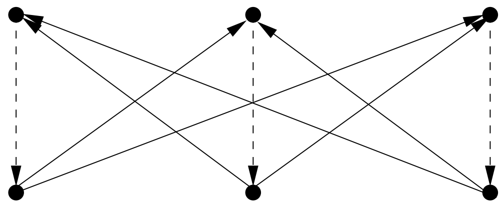
$$P^n_{LO} := conv\{\chi^T \in \{0,1\}^n \mid T \subseteq A_n \text{ is an acyclic tournament}\}.$$

- The vertices of P^n_{LO} correspond to acyclic tournaments.
- To solve the linear ordering problem minimize a linear function over this polytope, where an objective coefficient c_{ij} corresponds to the arc ij. In the example c_{ij} would be the number of comparisons ranking object O_i before object O_j.
- For algorithmic purposes a description of the polytope as a convex hull is of no help. Instead we need a description by a system of linear equations $Dx = d$ and linear inequalities $Ax \leq b$. In addition we are interested in finding a minimal equation system and a nonredundant system of inequalities.
- An integer formulation of the linear ordering problem is

$$\min_{\pi \in S(n)} \sum_{ij \in A_n} c_{ij} x_{ij}$$

$$\begin{aligned}
\text{s.t.} \quad & x_{ij} + x_{ji} = 1 && \text{for all } i,j \in V_n, i<j, && (1)\\
& x_{ij} + x_{jk} + x_{ki} \leq 2 && \text{for all } i,j,k \in V_n, i<j, i<k, j \neq k, && (2)\\
& x_{ij} \leq 1 && \text{for all } i,j \in V_n, i<j, && (3)\\
& x_{ij} \in \{0,1\} && \text{for all } i,j \in V_n, i<j. && (4)
\end{aligned}$$

- Equations (1) give a minimal equation system, inequalities (2) are *3-dicycle inequalities* and comprise, together with equations (1), all other cycle inequalities. Inequalities (3) are the trivial *hypercube inequalities*.
- For $n = 2, \dots, 5$ the linear ordering polytope P^n_{LO} is completely described by constraints (1)-(3).
- For $n \geq 6$ more constraints are needed as the following example shows, which is a non-integer vertex of the polytope defined by equalities (1) and the 3-cycle inequalities (2).

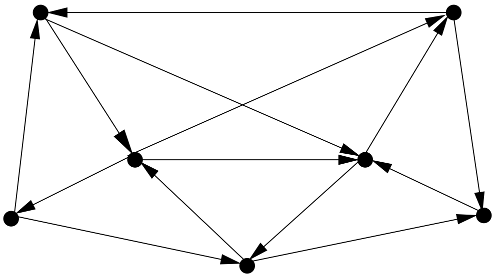


The values x_{ij} of the dashed edges are 0.5 and 1 for all others.

Research

- **Complete linear description** of the small linear ordering polytopes P^6_{LO} and P^7_{LO}.
- Design of a **branch-and-cut** algorithm for the linear ordering problem including two new separation routines.
  Both methods are based on the following concept. One combines known valid inequalities, for example 3-dicycle inequalities, such that in the resulting inequality all coefficients of the left hand side are divisible by k and the right hand side is not. There are two variants of this construction:
  - **Shortest Path Mod 2.** The advantage of this method is that there is no restriction on the constraints that are used for the generation of the new cut. On the other hand no extension from $k = 2$ to bigger prime numbers is possible and the violation of the resulting cuts is not maximal.
  - **Maximally violated mod-k.** In this method the augmentation to bigger prime numbers is relatively easy and the violation is maximal in any case. But only binding constraints can be used to generate a maximally violated mod-k cut.

  It is known that *Möbius ladders* and *3-fence inequalities* are contained in the class of mod-2 cuts, therefore no special separation routines for these constraint classes are necessary.
  Example of a Möbius ladder with 7 nodes:



- Development of **heuristics** for the determination of lower bounds for the linear ordering problem.

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