Graph Theoretic Characterization of Revenue Equivalence

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The key result in auction theory is the remarkable Revenue Equivalence Theorem. . .

Much of auction theory can be understood in terms of this theorem. . .

This talk

Characterization of RE via graph theory, not only for auctions

Introducing Revenue Equivalence: Single Item Auction

Bidders have valuation & utility

- n bidders
- bidder i has valuation v_i
  = "willingness to pay"
  ↖ private!
- looses $\Rightarrow$ utility is 0
- wins $\Rightarrow$ utility=valuation-price

Auction

- who will be the winner? allocation rule
- what will be the price per bidder? payment scheme

2nd Price Auction (Vickrey '61)

Allocation & payment rule

Bidders submit bids b_i by email

- allocate item to highest bid
- payment π_i = 2nd highest bid

Bidders strategy?

- truthtelling $b_i = v_i$, even if all other b_j known (i.e., truthtelling is a dominant strategy)

Result

Allocation rule is efficient (allocates to $v_{\max}$), auctioneer's revenue is (only) v_{n-1} . . . can we get more revenue?

1st Price Auction

Allocation & payment rule

Bidders submit bids b_i by email

- allocate item to highest bid
- payment $\pi_i = b_i$

Bidders strategy?

- trivial: bid below v_i (bid-shading), but by how much? (now depends on given information on other bidders!)

Result

Allocation rule is efficient (allocates to $v_{\max}$), to compare (expected) revenues, look at simple example. . .

Two Auctions: Revenues

- assume 2 bidders only
- both valuations v_j are i.i.d., uniform on $[0, 1]$

2nd price auction (Vickrey)

- bid $b_j = v_j$ (dominant strategy equilibrium)
- revenue collected $E[\min\{v_1, v_2\}] = \frac{1}{3}$

1st price auction

- bid $b_j = \frac{n-1}{n} v_j = \frac{1}{2} v_j$ (Bayes-Nash equilibrium)
- revenue collected $\frac{1}{2} E[\max\{v_1, v_2\}] = \frac{1}{2}\left(\frac{2}{3}\right) = \frac{1}{3}$

Auctions are quite different, expected revenues are equivalent

Revenue Equivalence (RE)

- auctioning a single item
- bidders uncertain about other bidders' valuations

Textbook Theorem

Suppose bidders'valuations are i.i.d. and bidders are risk neutral (maximizing expected utility). Then any [. . .] standard auction[a] yields the same (expected) revenue to the seller.

Example: 1st price auction $\leftrightarrow$ 2nd price auction

[a] Efficient: bidder with $v_{\max}$ wins
Individual rational: losers pay 0

see: Vickrey '61/'62, Riley & Samuelson '81, Myerson '81

Revenue Equivalence — Consequences

As auction designer

- given some auction with (expected) revenue X
- natural approach to increase X: optimize the payments
- but, whenever revenue equivalence holds ... to increase revenue need to modify the allocation rule

Example

Using 'reserve prices' in auctions increases expected revenue (at the expense of possibly not allocating the item)

Mechanism Design: Setting

- agents $i = 1, \dots, n$
- types $t_i \in T_i$, private information
- outcomes $a \in A$
- valuations $v_i : A \times T_i \to \mathbb{R}$, (or: $v_i : T \to \mathbb{R}^A$)

Direct revelation mechanism

given reports $t_1, \dots, t_n$ of all agents

mechanism: (f, π)

allocation rule $f(t_1, \dots, t_n) = a$

payment scheme $\pi_i(t_1, \dots, t_n) \in \mathbb{R}$ payment from i

utility = valuation - payment, $u_i = v_i(f(t), t_i) - \pi_i(t)$

Concepts

Definition (truthful mechanism)

(f, π) truthful iff for all agents i, reports $t_{-i} = (\dots, t_{i-1}, t_{i+1}, \dots)$,

utility from truth-telling $t_i \geq$ utility from lying s_i

$\rightarrow$ allocation rule f is called (truthfully) implementable

Why care about truthfulness?

By Myerson's revelation principle, this restriction is w.l.o.g.

Definition (revenue equivalence, RE)

Let f truthfully implementable. f satisfies RE iff for all truthful (f, π) and (f, π'), for all agents i, $\pi_i - \pi'_i = const. \; \forall \; t_{-i}$

Revenue Equivalence: Literature

I

Sufficient conditions on agents' preferences (T, v)

- (Green+Laffont '77, Holmström '79):
  f = utilitarian maximizer
- (Myerson '81, Krishna+Maenner '01, Milgrom+Segal '02):
  all implementable f

II

Characterization of agents' preferences (T, v)

- (Suijs '96):
  on finite outcome spaces, f = utilitarian maximizer satisfies RE
- (Chung+Olszewski '07):
  on finite outcome spaces, *all* implementable f satisfy RE

III

Our result

characterize agents preferences (T, v) *and* f s.t. RE holds, arbitrary outcome space

Link to Graph Theory: Allocation Graph G_f

fix one agent i and reports t_{-i} of others (notation: drop index i)

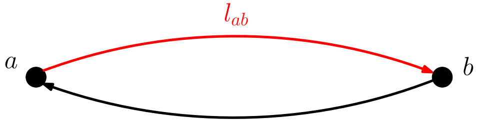
Allocation graph G_f for agent i

complete directed graph

- node set: possible outcomes $a, b \in A$ (may be infinite)
- arc lengths

$$\ell_{ab} = \inf_{t \in f^{-1}(b)} [v(b,t) - v(a,t)]$$

"if true type is any t with $f(t) = b$, $\ell_{ab} =$ (least) gain in valuation for truthtelling instead of lying to get outcome a"



Node Potentials

Remark: Payments for outcomes

- (f, π) truthful and $f(s) = f(t) = a$ for two reports s and t, then $\pi(s) = \pi(t)$

$\Rightarrow$ w.l.o.g. define payments $\pi(a)$ for outcomes $a \in A$ only

Definition (node potential)

$\pi : G_f \longrightarrow \mathbb{R}$ such that (shortest path) $\triangle$-inequality holds for all arcs (a, b):

$$\pi(b) \leq \pi(a) + \ell_{ab}$$

Truthful Mechanism $\Leftrightarrow$ Node Potential

Observation (Rochet, 1987)

(f, π) truthful $\Leftrightarrow \pi(\cdot)$ node potential in G_f

(f, π) truthful iff for any outcomes a, b:

utility truth-telling $t \in f^{-1}(b) \geq$ utility lying false $s \in f^{-1}(a)$

$\Leftrightarrow v(b, t) - \pi(b) \geq v(a, t) - \pi(a) \qquad \forall t \in f^{-1}(b)$

$\Leftrightarrow \pi(a) + [v(b, t) - v(a, t)] \geq \pi(b) \qquad \forall t \in f^{-1}(b)$

$\Leftrightarrow \pi(a) + \inf_{t \in f^{-1}(b)}[v(b, t) - v(a, t)] \geq \pi(b)$

$\Leftrightarrow \pi(a) + \ell_{ab} \geq \pi(b)$ $\square$

Node Potentials

Observation

$$(f, \pi) \text{ truthful} \Leftrightarrow \pi \text{ node potential in } G_f$$

Consequence

$$f \text{ is implementable} \overset{\text{Rochet}'87}{\Leftrightarrow} G_f \text{ has node potential}$$
$$\overset{\text{well-known}}{\Leftrightarrow} G_f \text{ has no negative cycle}$$

Revenue equivalence?

f satisfies RE $\Leftrightarrow$ node potential in G_f unique (up to constant)

Unique Node Potential - Characterization

Proposition 1

Any two node potentials differ only by a constant

$$\Updownarrow$$

$$dist(v,w) + dist(w,v) = 0$$

Proof:

"$\Downarrow$" $dist(v,\cdot)$ and $dist(w,\cdot)$ are node potentials, so
$dist(v,w) = \underbrace{dist(w,w)}_{=0} + c$ and $\underbrace{dist(v,v)}_{=0} = dist(w,v) + c$

"$\Uparrow$" $\pi(w) - \pi(v) \leq dist(v,w)$ and $\pi(v) - \pi(w) \leq dist(w,v)$
so $\pi(w) = dist(v,w) + \pi(v)$, for all w
so $\pi(\,\cdot\,)$ and $dist(v,\cdot)$ differ by constant $\pi(v)$

Main Result: Characterization of RE

Theorem (Characterization of RE)

Truthfully implementable f satisfies revenue equivalence

$$\Updownarrow$$

For all outcomes a, b, $dist_{G_f}(a, b) + dist_{G_f}(b, a) = 0$

Proof.

- payment scheme $\pi \Leftrightarrow$ node potential in G_f
- $dist_{G_f}(a, b) + dist_{G_f}(b, a) = 0$ necessary and sufficient condition for unique node potential in G_f ($\pm$ constant)

$\square$

Application I: Sufficient Conditions for RE

Theorem 1 (A finite)

- agents' types T (topologically) connected
- for all $a \in A$, valuations $v(a, \cdot)$ continuous on T

Then any truthfully implementable f satisfies revenue equivalence

Theorem 2 (A infinite, countable)

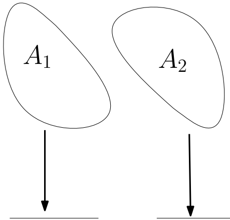
- agents' types $T \subseteq \mathbb{R}^k$, (topologically) connected
- valuations $v(a, \cdot)$ equicontinuous on T

Then any truthfully implementable f satisfies revenue equivalence

Theorems 1 and 2 aren't new – yet had heavier proofs before

Proof Idea (A finite)

Pick any partition of A:



T connected: $t \in \overline{f^{-1}(A_1)} \cap \overline{f^{-1}(A_2)}$ partition of T

A finite
v continuous
f truthful

$$\exists a_1 \in A_1, a_2 \in A_2 : \mathit{dist}(a_1, a_2) + \mathit{dist}(a_2, a_1) = 0$$

Exercise: sufficient for $\mathit{dist}(a, b) = \mathit{dist}(b, a)$ in G_f. $\square$

Application II: Demand Rationing

Setting

- distribute 1 unit of divisible good among n agents
- agent i has demand $t_i \in (0,1]$, $f_i =$ amount allocated to i,
- $v_i(f_i, t_i) = \begin{cases} 0, & \text{if } f_i \geq t_i; \\ f_i - t_i, & \text{if } f_i < t_i. \end{cases}$

Dictatorial allocation rule

Let $f_1 = t_1$, split rest equally among agents $2, \dots, n$

- this f is implementable
- but RE doesn't hold: $\pi_1(t) = 0$ and $\pi_1'(t) = t_1 - 1$ are both truthful for agent 1

$\Rightarrow$ All known results ("..., all implementable f satisfy RE") silent!

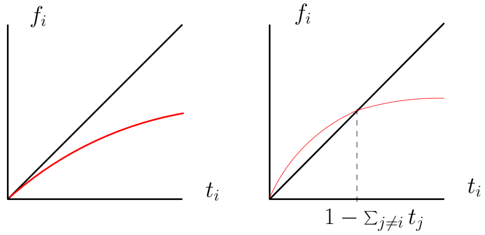
Proportional Rule

Can show: The proportional rule

$$f_i(t) = t_i / (\sum_{j=1}^{n} t_j) \text{ is implementable \& satisfies RE}$$

fixing t_{-i}, the 'report-outcome function' $f_i(t_i)$ is one of the cases

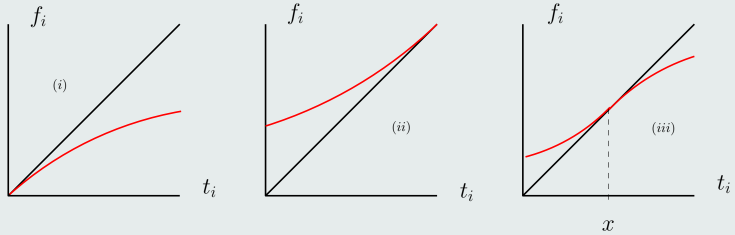
$\boxed{\sum_{j\neq i} t_j \geq 1}$ $\boxed{\sum_{j\neq i} t_j < 1}$



Demand Rationing: RE

Theorem

If report-outcome functions $f_i(t_i)$ are continuous, and any of cases (i), (ii) or (iii) holds for every agent i (and t_{-i}), then RE holds.



Proof.

Explicitly compute *dist* functions in G_f and case distinction - tedious but not too hard □

Literature Comparison - Bottom Line

previous characterizations

- Suijs '96 is a special case of ours
- Chung & Olszewski (C&O '07) can be derived quite easily

previous sufficient conditions

- Green+Laffont '77
- Holmström '79
- Krishna+Maenner '01
- Milgrom+Segal '02

can be derived, too (as also done by C&O '07)

Summary

- simple(!) characterization of RE, graph theory is key
- first condition on preferences *and* allocation rule together

  applies also in settings, where all previous results are silent
- works same way for other equilibrium concepts

  Bayes-Nash, Ex-post with externalities

- Myerson, R. (1981). Optimal auction design. *Mathematics of Operations Research* **6**, 58-73.

- Heydenreich, Müller, Uetz, Vohra (2009). Characterization of revenue equivalence. *Econometrica* **77**, 307-316

both are online