A Polyhedral Study of the Time-dependent Traveling Salesman Problem

Hernán Abeledo

George Washington University, USA

Artur Pessoa & **Eduardo Uchoa**

Universidade Federal Fluminense, Brazil

Ricardo Fukasawa

IBM Research, USA

The Time-dependent Traveling Salesman Problem (TDTSP)

Generalization of the Asymmetric Traveling Salesman Problem (ATSP), where arc costs depend on their position in the tour with respect to a chosen start node.

- Instead of arc costs $c(i,j)$, there are arc-position costs $c(i,j,t)$.
- The positions are interpreted as “times”, each visit takes a unit of time.

Motivation I

The TDTSP is an interesting problem with a number of applications in routing and scheduling. Examples:

- The Traveling Deliveryman Problem (a.k.a. Minimum Latency Problem),
- The $1|s_{ij}|\Sigma Cj$ scheduling problem.

Motivation II

The TDTSP can be generalized to allow multiple routes and “non-unitary” times. Many classical routing and scheduling problems can be represented in that way.

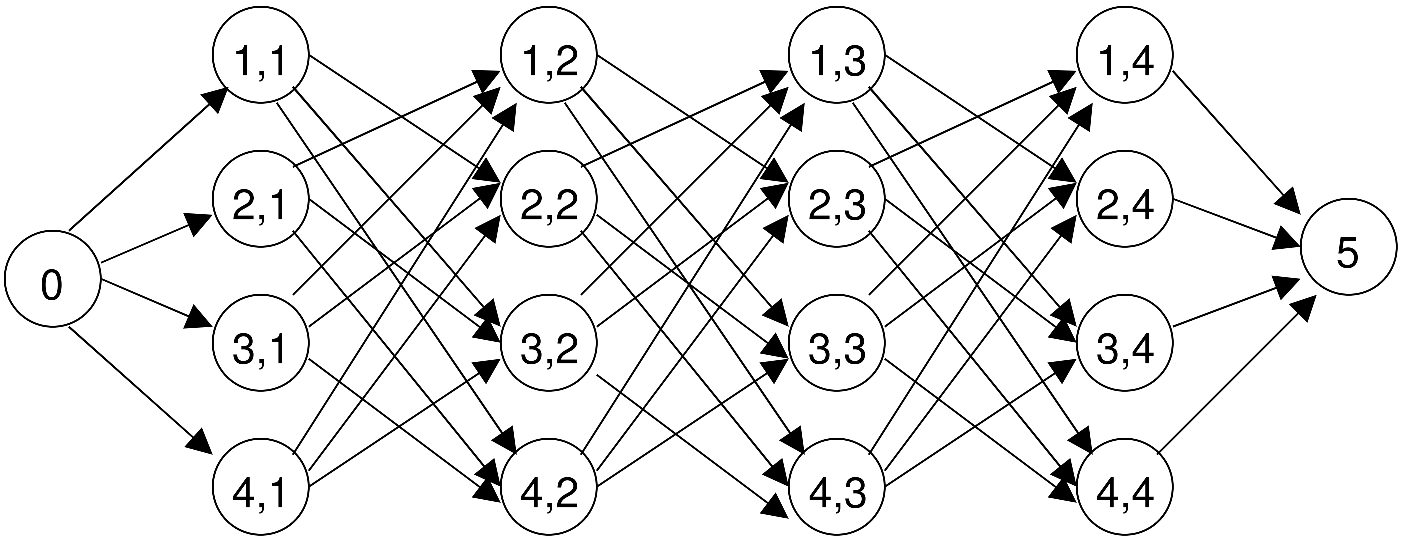
- The new TDTSP facets found can be generalized to effective valid inequalities (not proved to define facets) for those cases.

Motivation III

Known STSP and ATSP facets define disappointingly low dimensional faces of the TDTSP polytope.

- Perhaps, it can be possible to combine and project TDTSP facets into new STSP and ATSP facets.

Picard & Queyranne Formulation (1978)



Set of original nodes $N_0 = \{0,\ldots,4\}$, T=5

Picard & Queyranne Formulation (1978)

- Let $N_0=\{0,1,\ldots,n\}$, $N=\{1,\ldots,n\}$ and $N(i)=N-\{i\}$.
- PQ: constrained shortest path from 0 to node T: for each $v\in N$, exactly one (v,j) must be visited .
- Notation:
  - (i,j,t) denotes an arc from node (i,t) to node (j,t+1).
  - $x_{i,j}^{t}$ represents the flow on arc (i,j,t)

Picard & Queyranne Formulation (1978)

$$\text{Minimize} \sum_{j \in N} c_{0,j}^0 \, x_{0,j}^0 + \sum_{t=1}^{n-1} \sum_{i \in N} \sum_{j \in N(i)} c_{i,j}^t \, x_{i,j}^t + \sum_{i \in N} c_{i,T}^n \, x_{i,T}^n$$

S.t.

$$\sum_{j \in N} x_{0,j}^0 = 1$$

$$x_{0,j}^0 = \sum_{k \in N(j)} x_{j,k}^1, \; j = 1 \ldots n$$

$$\sum_{i \in N(j)} x_{i,j}^t = \sum_{k \in N(j)} x_{j,k}^{t+1}, \; j = 1 \ldots n, t = 1 \ldots n-2$$

$$\sum_{i \in N(j)} x_{i,j}^{n-1} = x_{j,T}^n, \; j = 1 \ldots n$$

$$x_{0,j}^0 + \sum_{t=1}^{n-1} \sum_{i \in N(j)} x_{i,j}^t = 1, \; j = 1 \ldots n$$

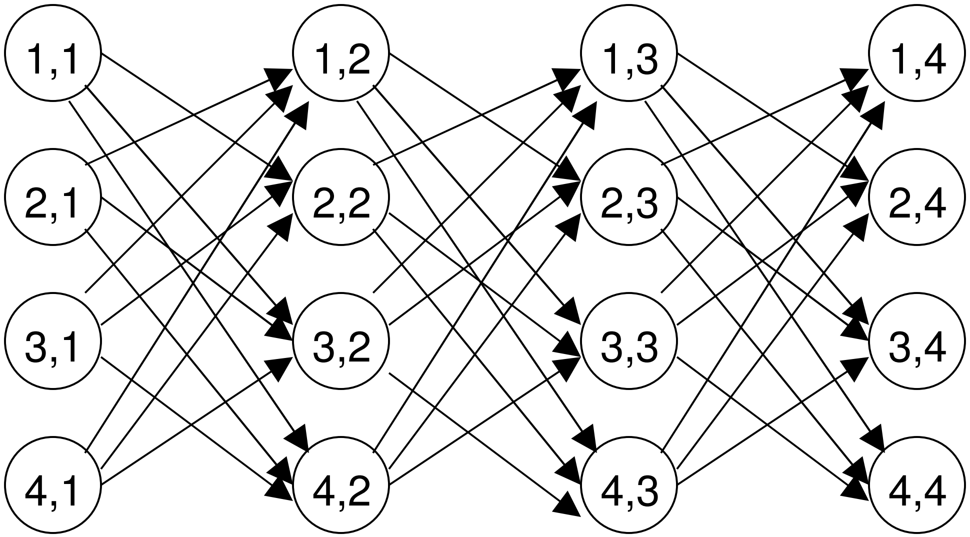
$$x \geq 0$$

x integer

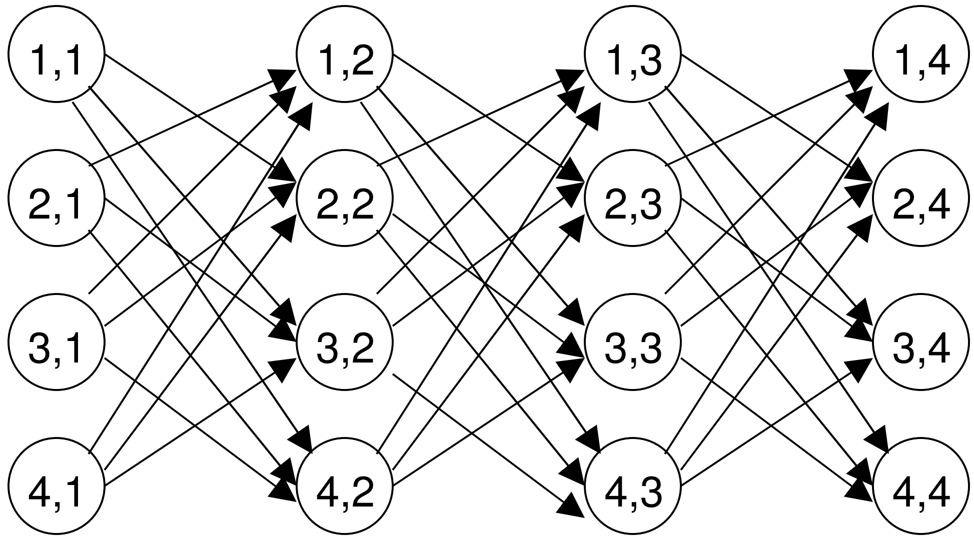
Picard & Queyranne Formulation (1978)

- Large size, $O(n^3)$ vars and $O(n^2)$ constraints.
- Dantzig-Wolfe decomposition produces an equivalent path formulation with n constraints (the degree constraints) and an exponential number of path variables, that can be priced in $O(n^3)$ time.
- Not really strong. Bounds on the classical STSP inferior to DFJ(54).
- Our Goal: study the polyhedral properties of the TDTSP, identifying strong valid inequalities on a slightly modified PQ formulation that is more convenient for that mathematical study.

Modified formulation



Eliminate 2n variables incident to nodes 0 and T
(Their costs can be included in the arcs in the first and last layer)



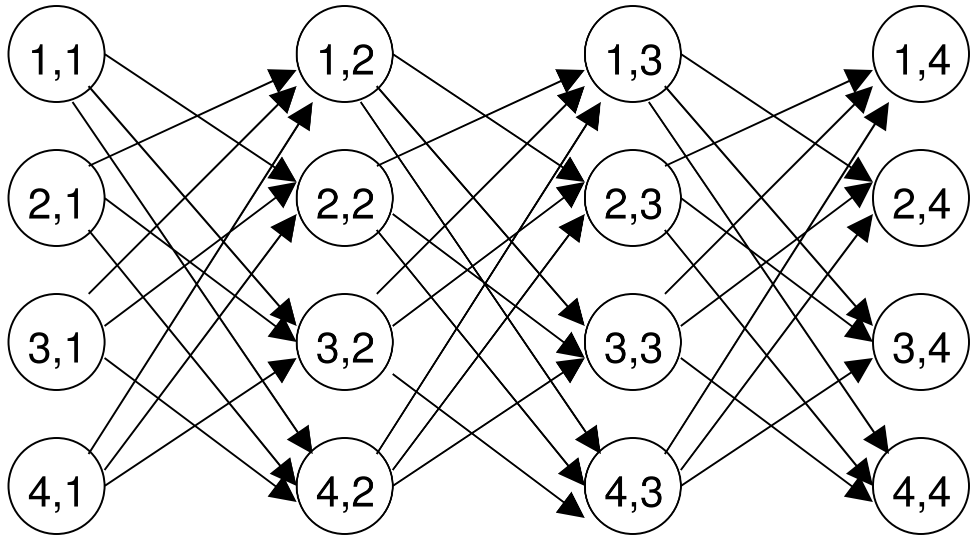
One arc in the first layer

$$\sum_{i\in N}\sum_{j\in N(i)} x^1_{i,j} = 1$$

$$\sum_{i\in N(j)} x^t_{i,j} = \sum_{k\in N(j)} x^{t+1}_{j,k},\ j = 1\ldots n, t = 1\ldots n-2$$

$$\sum_{k\in N(j)} x^1_{j,k} + \sum_{t=1}^{n-1}\sum_{i\in N(j)} x^t_{i,j} = 1,\ j = 1\ldots n$$

$x \geq 0$ and integer



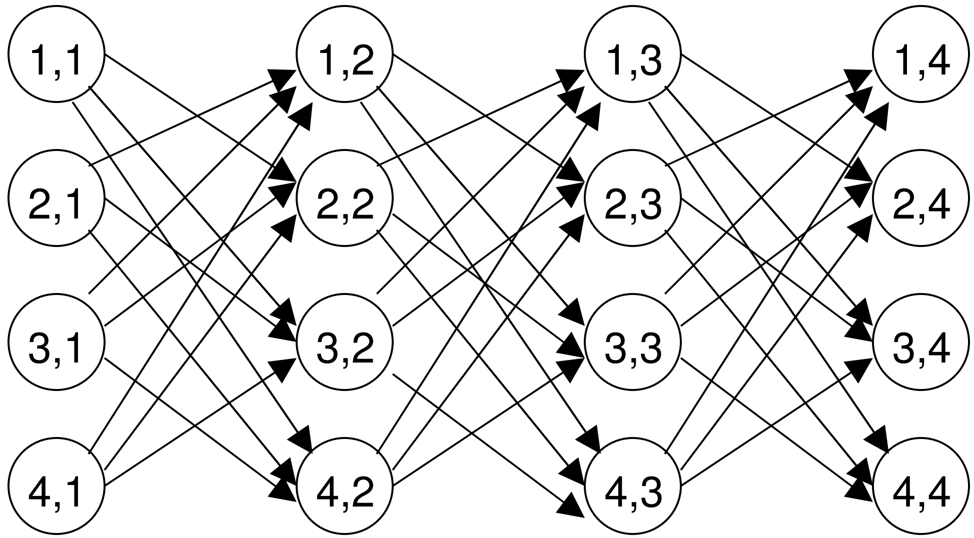
Flow conservation

$$\sum_{i \in N} \sum_{j \in N(i)} x^1_{i,j} = 1$$

$$\sum_{i \in N(j)} x^t_{i,j} = \sum_{k \in N(j)} x^{t+1}_{j,k}, \; j = 1 \ldots n, t = 1 \ldots n-2$$

$$\sum_{k \in N(j)} x^1_{j,k} + \sum_{t=1}^{n-1} \sum_{i \in N(j)} x^t_{i,j} = 1, \; j = 1 \ldots n$$

$x \geq 0$ and integer



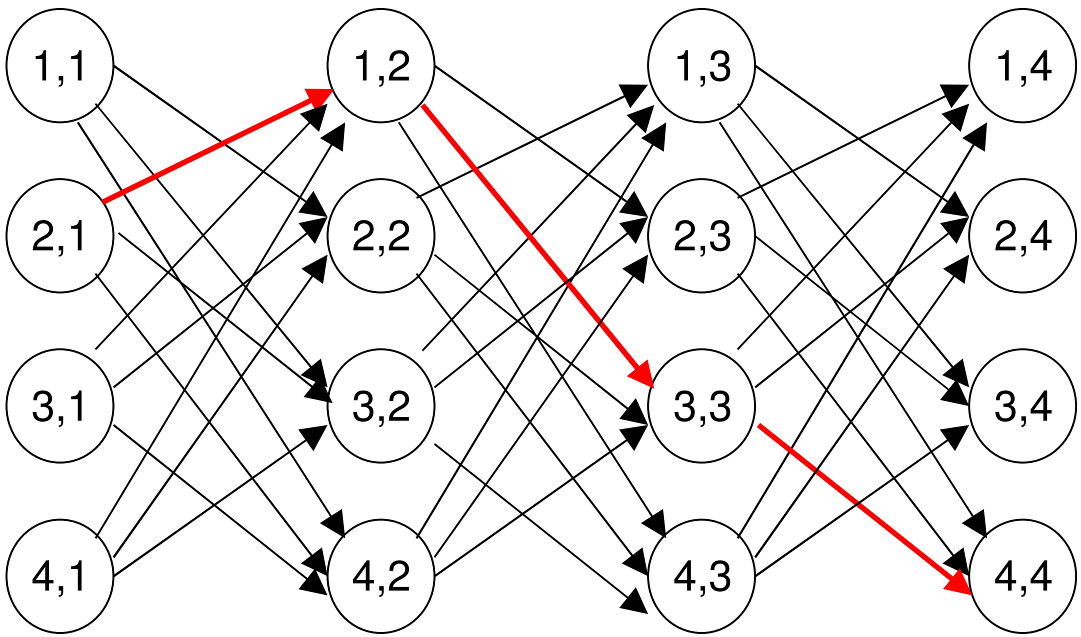
degree constraints

$$\sum_{i \in N} \sum_{j \in N(i)} x_{i,j}^1 = 1$$

$$\sum_{i \in N(j)} x_{i,j}^t = \sum_{k \in N(j)} x_{j,k}^{t+1}, \ j = 1 \ldots n, t = 1 \ldots n-2$$

$$\sum_{k \in N(j)} x_{j,k}^1 + \sum_{t=1}^{n-1} \sum_{i \in N(j)} x_{i,j}^t = 1, \ j = 1 \ldots n$$

$x \geq 0$ and integer



Solution 0->2->1->3->4->0

Modified formulation

$$\sum_{i \in N} \sum_{j \in N(i)} x^1_{i,j} = 1$$

$$\sum_{i \in N(j)} x^t_{i,j} = \sum_{k \in N(j)} x^{t+1}_{j,k}, \; j = 1 \ldots n, t = 1 \ldots n-2$$

$$\sum_{k \in N(j)} x^1_{j,k} + \sum_{t=1}^{n-1} \sum_{i \in N(j)} x^t_{i,j} = 1, \; j = 1 \ldots n$$

$$x \geq 0 \text{ and integer}$$

- $n(n-1)^2$ variables and $n^2 - n + 1$ equations
- P_n = convex hull of integer solutions
- Lemma: System of equations has rank $n^2 - n$

Polytope dimension

- **Theorem:** let $n \geq 5$, then

$$\dim (P_n) = n(n-1)(n-2)$$

proof: by induction on n

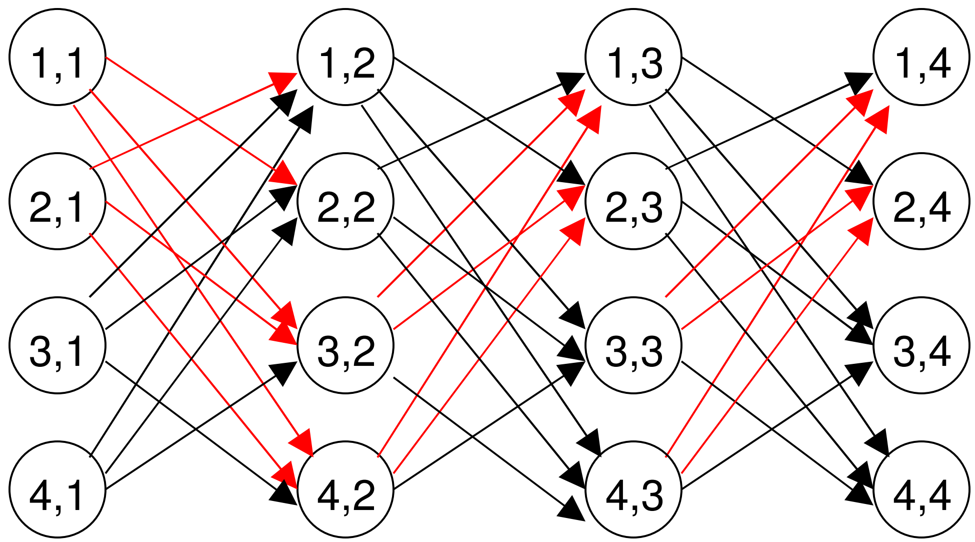
ATSP Subtour Elimination Cuts

- On the ATSP arc formulation:

$$\sum_{i \notin S} \sum_{j \in S} x_{i,j} \geq 1, \quad \forall S \subset N, 1 < |S| < n$$

- On our TDTSP formulation:

$$\sum_{i \in S} \sum_{j \in N(i)} x_{i,j}^{1} + \sum_{t=1}^{n-1} \sum_{i \notin S} \sum_{j \in S} x_{i,j}^{t} \geq 1, \quad \forall S \subset N, 1 < |S| < n$$



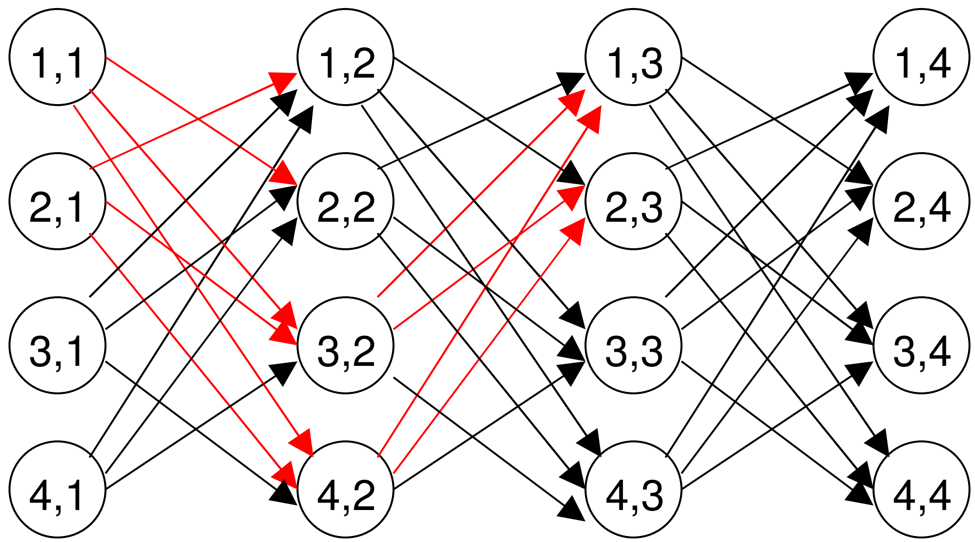
- If S={1,2}

$$\sum_{i\in S}\sum_{j\in N(i)} x_{i,j}^{1} + \sum_{t=1}^{n-1}\sum_{i\notin S}\sum_{j\in S} x_{i,j}^{t} \geq 1, \quad \forall S\subset N, 1<|S|<n$$

Lifted Subtour Elimination Cuts

- Both inequalities state that at least one arc must enter S. However, on the TDTSP the inequality can be made much stronger:

$$\sum_{i \in S} \sum_{j \in N(i)} x_{i,j}^{1} + \sum_{t=1}^{n-|S|} \sum_{i \notin S} \sum_{j \in S} x_{i,j}^{t} \geq 1, \quad \forall S \subset N, 1 < |S| < n$$



- S={1,2}

$$\sum_{i\in S}\sum_{j\in N(i)} x_{i,j}^{1} + \sum_{t=1}^{n-|S|}\sum_{i\notin S}\sum_{j\in S} x_{i,j}^{t} \geq 1, \quad \forall S\subset N, 1<|S|<n$$

Lifted Subtour Elimination Cuts

- At least one arc with index less or equal to n-|S| must enter S, otherwise the route can not "cover" S.

$$\sum_{i \in S} \sum_{j \in N(i)} x_{i,j}^{1} + \sum_{t=1}^{n-|S|} \sum_{i \notin S} \sum_{j \in S} x_{i,j}^{t} \geq 1, \quad \forall S \subset N, 1 < |S| < n$$

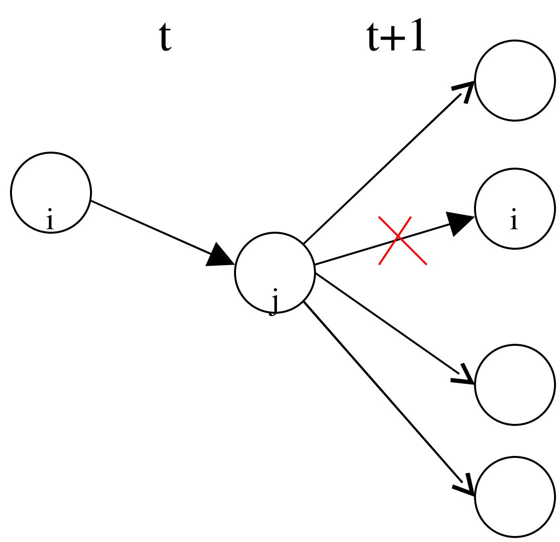
Lifted Subtour Elimination Cuts

- **Theorem:** If $n \geq 6$ and $2 < |S| < n-1$ then a lifted subtour elimination cut defines a facet of P_n.

proof: by induction on n and |S|.

Two-cycle elimination Cuts

- A unit flow on arc (i,j,t) must exit node (j,t+1) using arcs other than (j,i,t+1)



- **Theorem**: for n $\geq$ 6, $x_{i,j}^{t} \leq \sum_{k \in N \setminus \{i,j\}} x_{j,k}^{t+1}$ defines a facet of P_n.

Flow decomposition

- **Theorem**: Let $x \geq 0$ be feasible for

$$\sum_{j \in N} x^0_{0,j} = 1$$
$$x^0_{0,j} = \sum_{k \in N(j)} x^1_{j,k},\ j = 1 \ldots n$$
$$\sum_{i \in N(j)} x^t_{i,j} = \sum_{k \in N(j)} x^{t+1}_{j,k},\ j = 1 \ldots n, t = 1 \ldots n-2$$
$$\sum_{i \in N(j)} x^{n-1}_{i,j} = x^n_{j,T},\ j = 1 \ldots n$$

<u>and</u> $x^t_{i,j} \leq \sum_{k \in N \setminus \{i,j\}} x^{t+1}_{j,k}$ for all $i,j \in N$, $1 \leq t < n\text{-}1$

then x can be decomposed as path flows from 0 to T, each with no two-cycles.

Flow decomposition

- Idea of the proof: transform each node into a transportation problem and use results by Gale (57) and Hoffman (60) characterizing feasible network flow problems.

Flow decomposition and the path formulation

One can strength the PQ bound by eliminating s-cycles in the equivalent path formulation.

- 2-cycle elimination is algorithmically simple and does not change the pricing complexity of $O(n^3)$.
- s-cycle elimination, for $s > 2$, is much harder and costs $O(s!s^2n^3)$ (IV03).

Flow decomposition and the path formulation

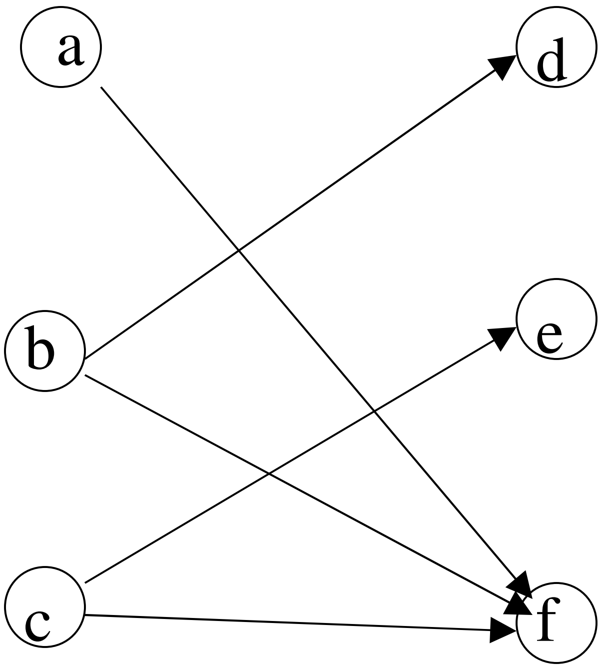
- Polyhedral 2-cycle elimination is also easy (two-cycle cuts suffice).
- Polyhedral s-cycle elimination, even for s=3, appears to be much harder.

Admissible flow constraints (generalize two-cycle elimination constraints)

Idea: the flow on a set of arcs entering a set of nodes cannot exceed the aggregate flow on a corresponding set of compatible outgoing arcs.

- Let *X* be a set of nodes *(i,t)* in the PQ formulation and *E* a set of arcs entering this set.
- For each arc *e* in *E*, *A(e)* is the set of arcs compatible with *a* and leaving *X*, i.e., *f* in *A(e)* iff there is a solution that uses *e* and leaves *X* for the first time in *f*.

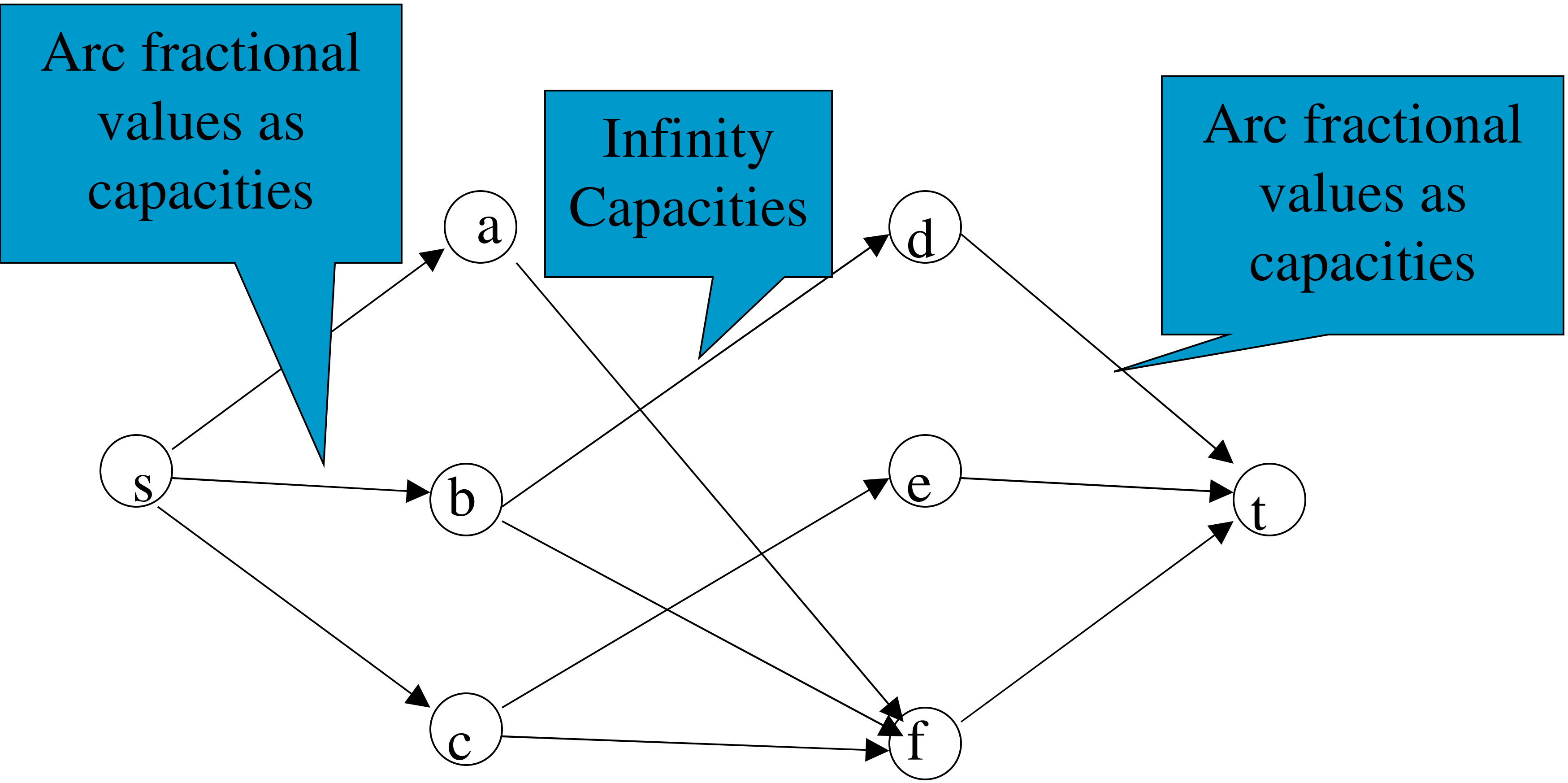
Admissible Flow Constraints



E={a,b,c},
A(a)={f},
A(b)={d,f}
A(c)={e,f}.

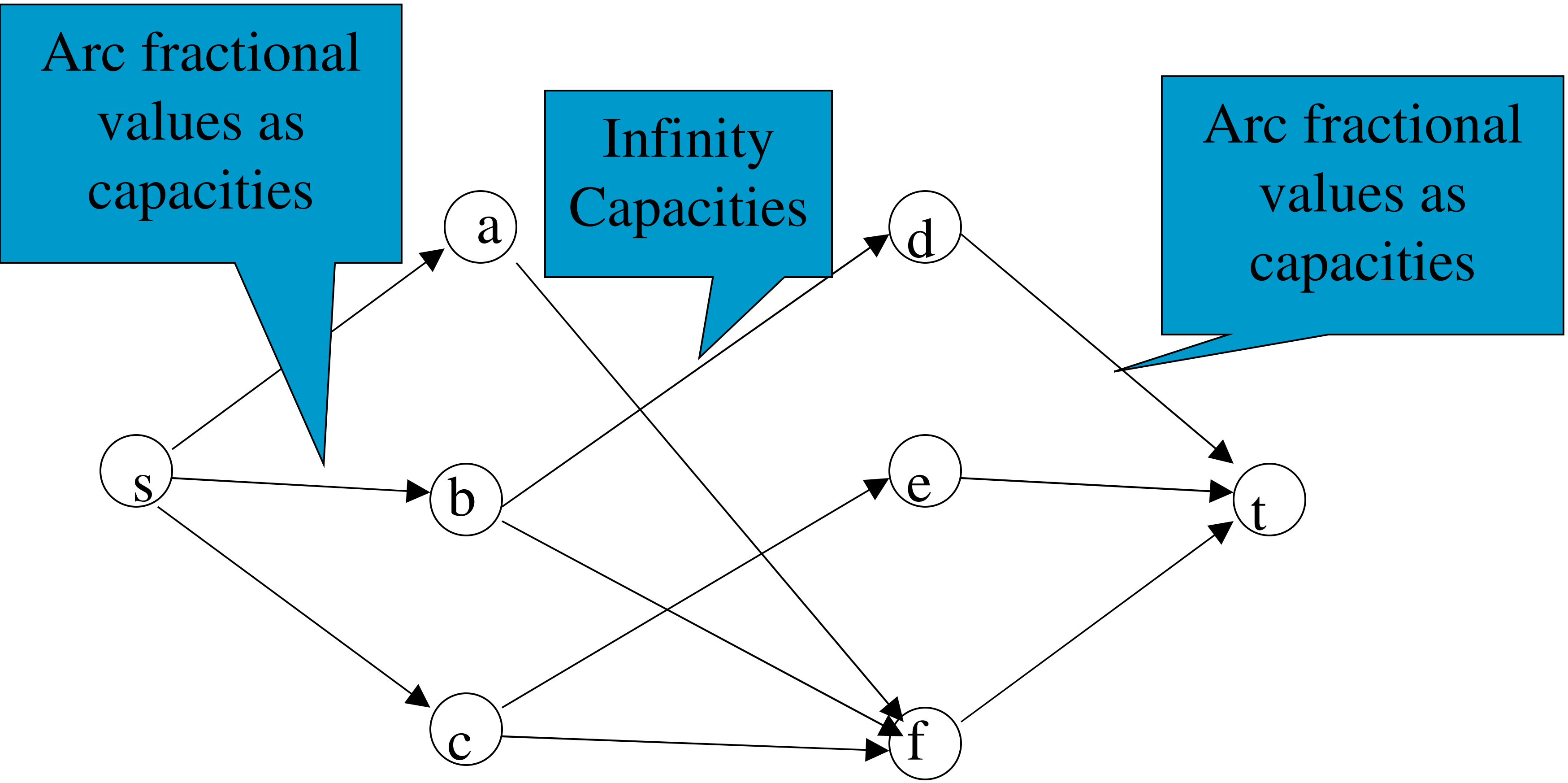
Set a bipartite graph indicating which entering arcs can correspond to each leaving arc

Admissible Flow Constraints



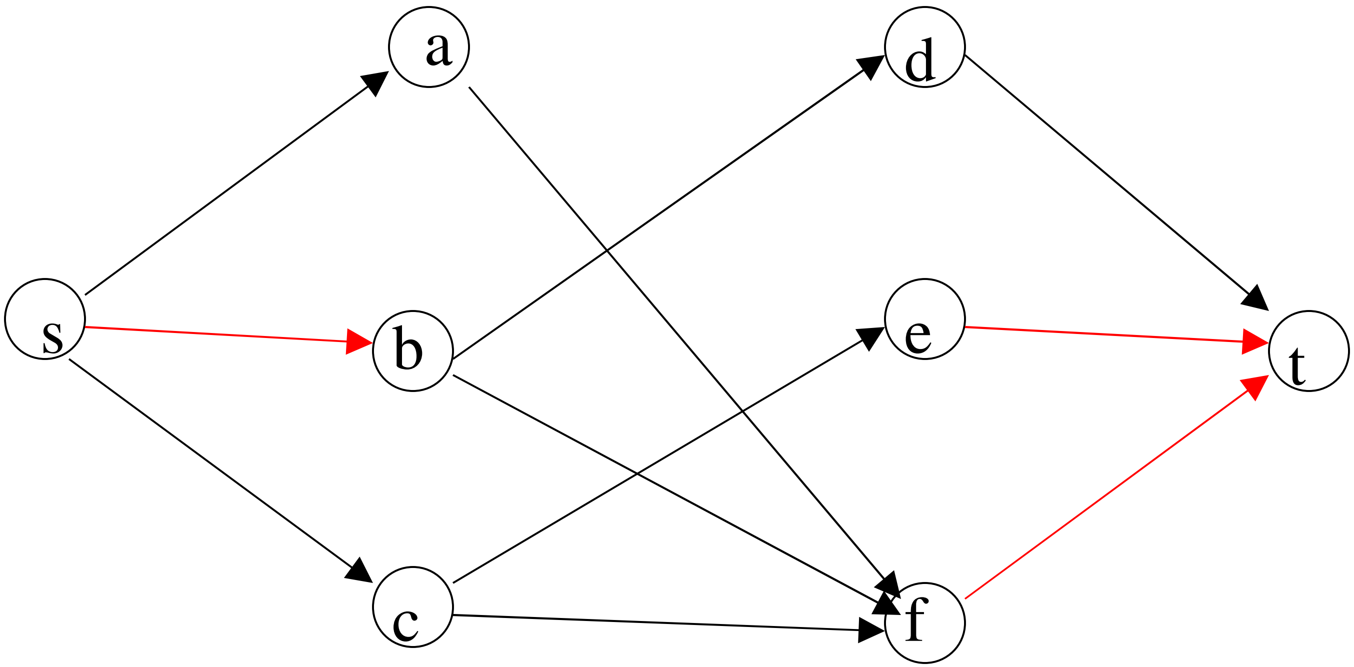
Solve this s-t max-flow min-cut problem

Admissible Flow Constraints



If the min-cut is smaller than the flow entering (and leaving) X, a violated cut is obtained

Admissible Flow Constraints



$$x_b + x_e + x_f >= x_a+x_b+x_c \;\; => x_e + x_f >= x_a+x_c$$

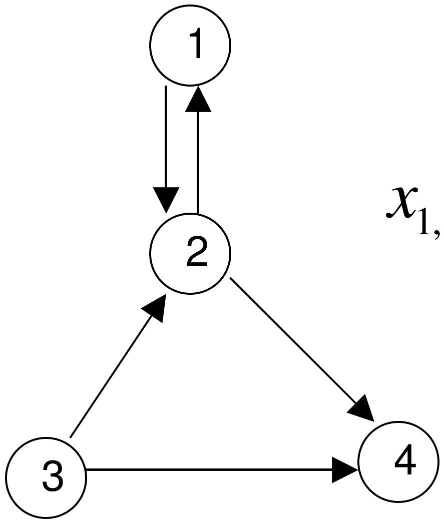
Admissible flow constraints

- Two cycle-elimination cuts are the particular case where X and E are unitary.
- **Theorem**: for $n \geq 6$, if X is a rectangular square of size two and E is unitary than the AFC defines a facet of P_n.

- AFCs appear to be strong cuts.
- Hard to prove general facet results.

Cuts from the arc conflict graph

- ATSP CAT inequalites correspond to odd-holes in the arc conflict graph. Ex:



$$x_{1,2} + x_{3,2} + x_{3,4} + x_{2,4} + x_{2,1} \leq 2$$

Cuts from the arc conflict graph

- There are no clique cuts in the ATSP, for any 3 arcs not contained in an indegree or an outdegree constraint, there is at least one tour with 2 such arcs.

- The TDTSP arc-position variables produces a richer conflict graph and clique inequalities can be found.

TDTSP Cliques

There are two patterns in the TDTSP conflict graph producing interesting cliques

- Star cliques
- Triangle cliques

TDTSP Cliques

There are two patterns in the TDTSP conflict graph producing interesting cliques

- Star cliques => define facets

The same facets already defined by the two-cycle elimination constraints! (P_n is not full-dimensional)

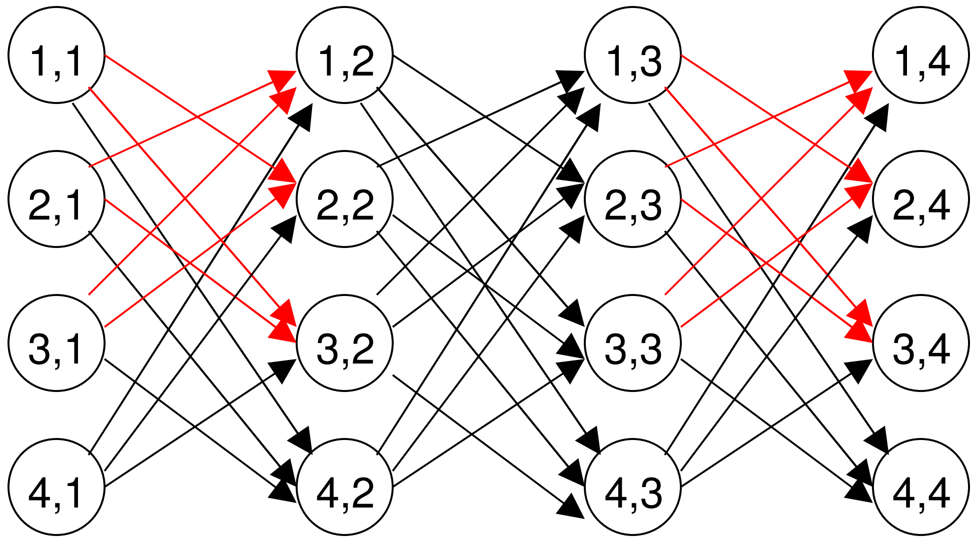
TDTSP Cliques

There are two patterns in the TDTSP conflict graph producing interesting cliques

- Triangle Cliques

Defined over sets S of size 3

Alternating Triangle Clique Cuts



- Ex: S={1,2,3}, $\sum_{i=1}^{3}\sum_{\substack{j=1 \\ j\neq i}}^{3} x_{i,j}^{1} + \sum_{i=1}^{3}\sum_{\substack{j=1 \\ j\neq i}}^{3} x_{i,j}^{3} \leq 1$

- **Theorem**: If n $\geq$ 6, the alternating triangle clique cuts define facets of P_n.

General Triangle Clique Cuts

- [Pessoa, Uchoa, Poggi (08)] showed that triangle clique cuts (alternating or not) are separable in polynomial time and used them succesfully to solve Heterogenous Vehicle Routing Problem instances.

- **Conjecture:** general triangle clique cuts are also facet defining.

Preliminary computational results

- Tests on the Traveling Deliveryman Problem Instances (minimize the average time to attend a set of n clients). Equivalent to the $1|s_{ij}|\Sigma Cj$ scheduling problem. The classical ATSP is equivalent to the $1|s_{ij}|C_{Max}$.

- Modelled as a TDTSP by setting

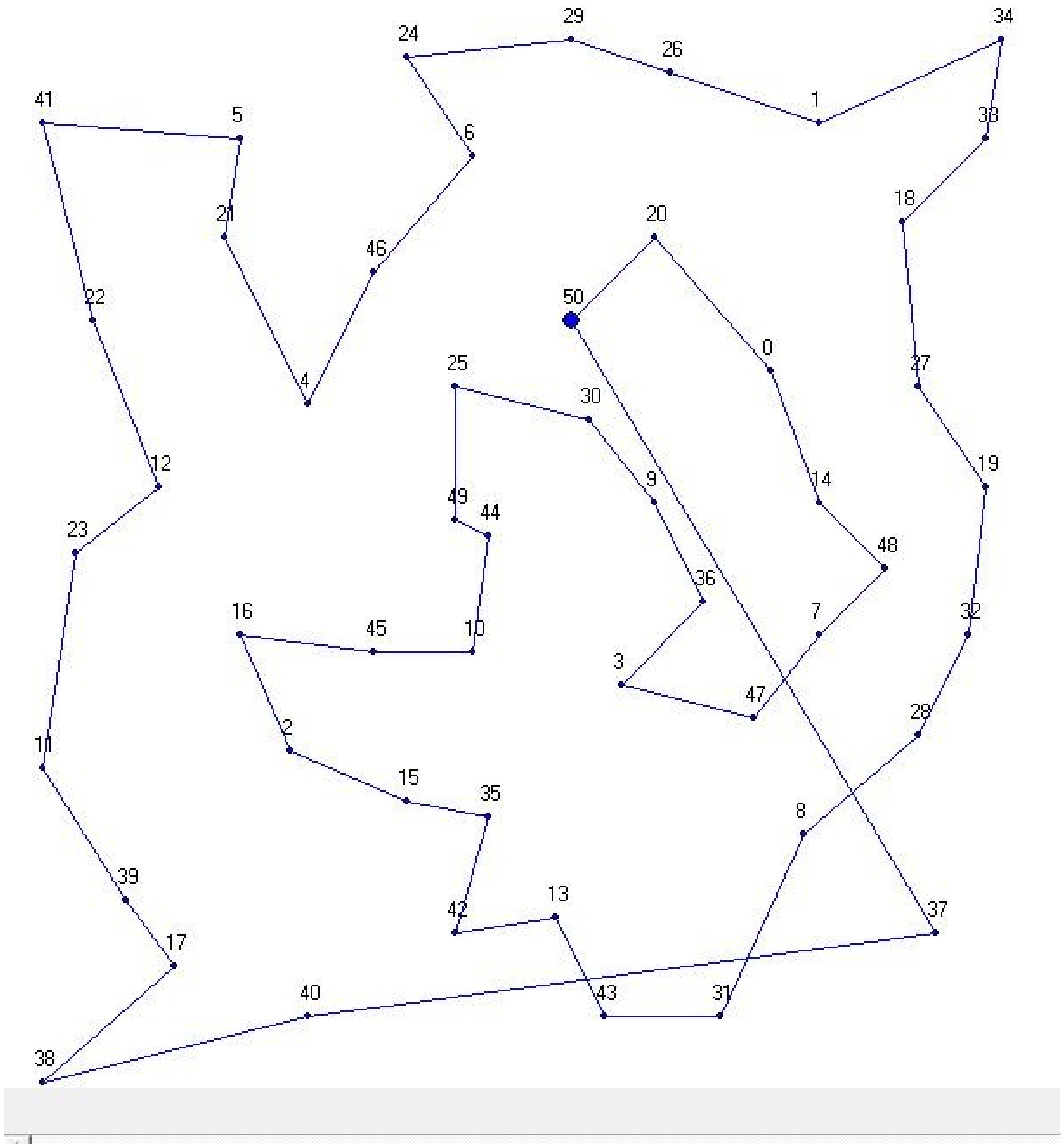
  $c(i,j,t) = (n-t)c(i,j)$

Traveling Deliveryman Problem

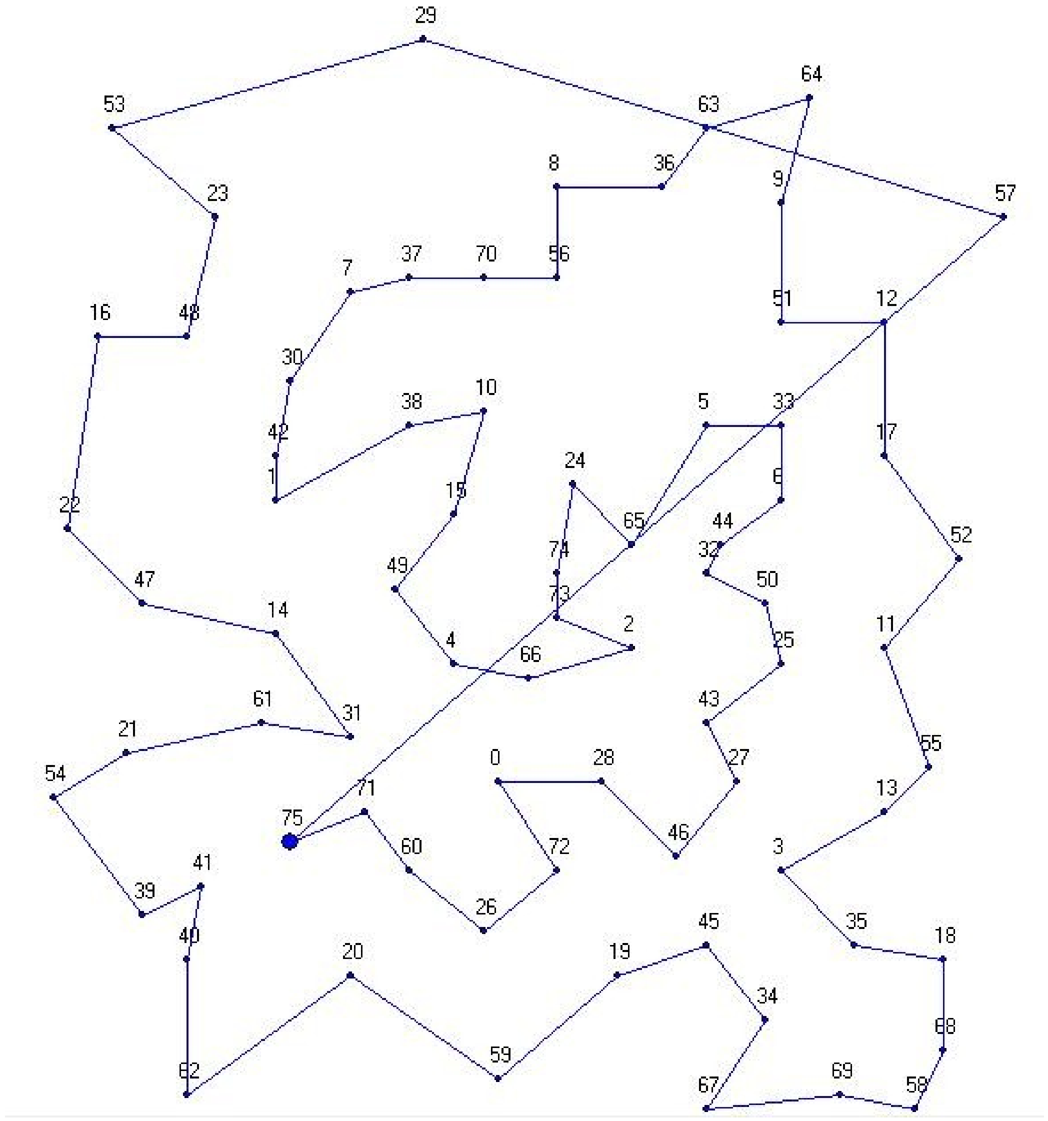
- Much harder than the classical TSP. Some recent works with good exact results

- Lucena (90)
- Fischetti, Laporte and Martello (94) – Combinatorial B&B, root gaps around 10%, very large seach trees, yet, could solve the largest instances (up to 60 vertices).
- Wu, Huang and Zhan (04)
- Bigras, Gamache and Savard (08) – Column generation + TSP cuts + cliques, root gaps around 1%.
- Sarubbi and Luna (08)
- Mendes (08)

Eil51 opt:10178



Eil76 opt: 17976



Preliminary computational results

- Lifted Subtour Elimination Cuts separated exactly by MIP (fast);
- Admissible Flow Constraints separated by solving min-cut problems over heuristically generated candidate sets;
- Triangle Cliques separated exactly in $O(n^3)$ time.

Preliminary computational results

- All the instances tested with up to 76 vertices were solved without branching.
- Large times (eil76 takes 10 hours).
- Current work: implementing the cuts in an efficient BCP code.

Thanks!