A new characterization of Seymour graphs

Zoltán Szigeti

Laboratoire G-SCOP
Grenoble, France

January 2009

joint with A. Ageev, A. Sebő

Outline

1. Motivation
2. Definitions : complete packing of cuts, joins
3. Seymour Graphs
4. Around Seymour graphs
5. Co-NP characterization of Seymour graphs
6. New Co-NP characterization of Seymour graphs
7. Proof
8. Algorithmic aspects
9. Open problem

Motivation

Edge-disjoint paths problem

Given a graph $H = (V, E)$ and k pairs of vertices $\{s_i, t_i\}$, decide whether there exist k edge-disjoint paths connecting the k pairs s_i, t_i.

Motivation

Edge-disjoint paths problem

Given a graph $H = (V, E)$ and k pairs of vertices $\{s_i, t_i\}$, decide whether there exist k edge-disjoint paths connecting the k pairs s_i, t_i.

Reformulation by adding the set F of edges $s_i t_i$.

Motivation

Edge-disjoint paths problem

Given a graph $H = (V, E)$ and k pairs of vertices $\{s_i, t_i\}$, decide whether there exist k edge-disjoint paths connecting the k pairs s_i, t_i.

Reformulation by adding the set F of edges $s_i t_i$.

Complete packing of cycles

Given a graph $H' = (V, E + F)$, decide whether there exist $|F|$ edge-disjoint cycles in H', each containing exactly one edge of F.

Motivation

Edge-disjoint paths problem

Given a graph $H = (V, E)$ and k pairs of vertices $\{s_i, t_i\}$, decide whether there exist k edge-disjoint paths connecting the k pairs s_i, t_i.

Reformulation by adding the set F of edges $s_i t_i$.

Complete packing of cycles

Given a graph $H' = (V, E + F)$, decide whether there exist $|F|$ edge-disjoint cycles in H', each containing exactly one edge of F.

Suppose H' is planar. The problem in the dual :

Motivation

Edge-disjoint paths problem

Given a graph $H = (V, E)$ and k pairs of vertices $\{s_i, t_i\}$, decide whether there exist k edge-disjoint paths connecting the k pairs s_i, t_i.

Reformulation by adding the set F of edges $s_i t_i$.

Complete packing of cycles

Given a graph $H' = (V, E + F)$, decide whether there exist $|F|$ edge-disjoint cycles in H', each containing exactly one edge of F.

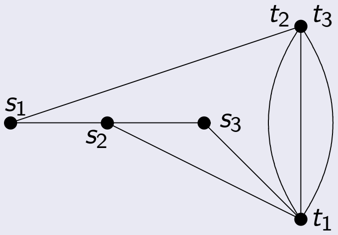
Suppose H' is planar. The problem in the dual :

Complete packing of cuts

Given a graph $G = (V', E' + F')$, decide whether there exist $|F'|$ edge-disjoint cuts in G, each containing exactly one edge of F'.

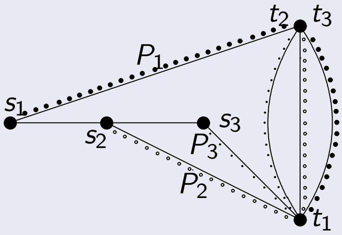
An example

Edge-disjoint paths problem



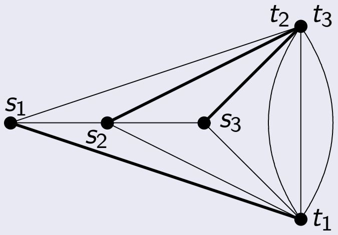
An example

Complete packing of paths



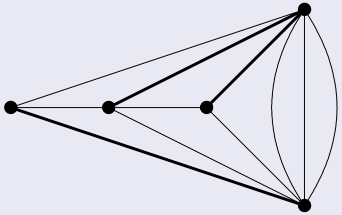
An example

Adding the edges



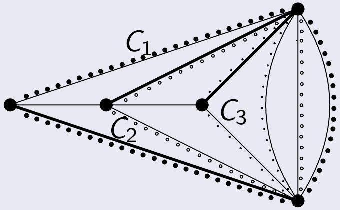
An example

The graph H'



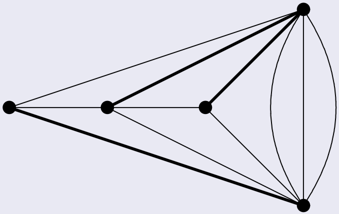
An example

Complete packing of cycles



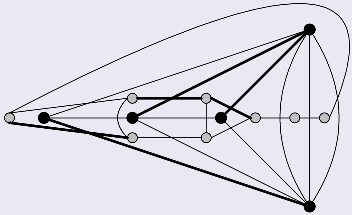
An example

H' is planar



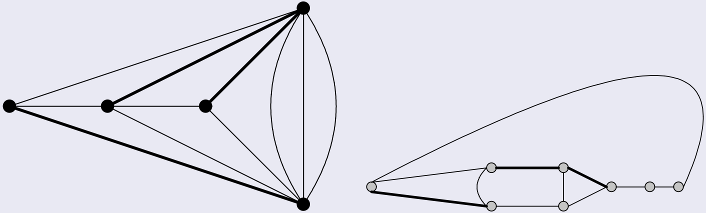
An example

H' and his dual



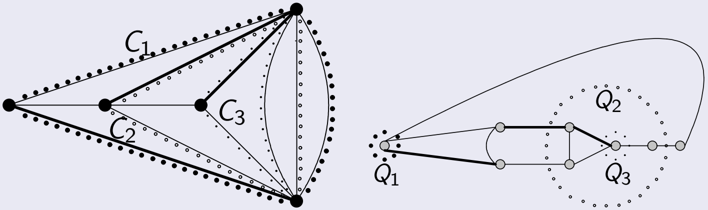
An example

H' and his dual



An example

Complete packing of cycles and cuts



Complete packing of cuts

The graphs are not planar anymore !

Complete packing of cuts

The problem

Given a graph $G = (V, E + F)$, decide whether there exist $|F|$ edge-disjoint cuts in G, each containing exactly one edge of F.

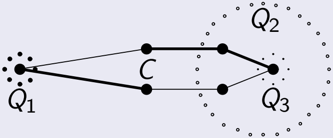
Complete packing of cuts

The problem

Given a graph $G = (V, E + F)$, decide whether there exist $|F|$ edge-disjoint cuts in G, each containing exactly one edge of F.

Necessary condition

If the graph $G = (V, E + F)$ admits a complete packing of cuts, then F is a join : for every cycle C, $|C \cap F| \leq |C \setminus F|$.



Complete packing of cuts

The problem

Given a graph $G = (V, E + F)$, decide whether there exist $|F|$ edge-disjoint cuts in G, each containing exactly one edge of F.

Necessary condition

If the graph $G = (V, E + F)$ admits a complete packing of cuts, then F is a join : for every cycle C, $|C \cap F| \leq |C \setminus F|$.

Sufficient condition ?

If F is a join, the graph $G = (V, E + F)$ admits a complete packing of cuts ?

Complete packing of cuts

The problem

Given a graph $G = (V, E + F)$, decide whether there exist $|F|$ edge-disjoint cuts in G, each containing exactly one edge of F.

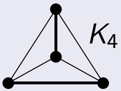
Necessary condition

If the graph $G = (V, E + F)$ admits a complete packing of cuts, then F is a join : for every cycle C, $|C \cap F| \leq |C \setminus F|$.

Sufficient condition ?

If F is a join, the graph $G = (V, E + F)$ admits a complete packing of cuts ?

NOT :



Complete packing of cuts

The problem

Given a graph $G = (V, E + F)$, decide whether there exist $|F|$ edge-disjoint cuts in G, each containing exactly one edge of F.

Necessary condition

If the graph $G = (V, E + F)$ admits a complete packing of cuts, then F is a join : for every cycle C, $|C \cap F| \leq |C \setminus F|$.

Sufficient condition ?

If F is a join, the graph $G = (V, E + F)$ admits a complete packing of cuts ?

Theorem (Middendorf, Pfeiffer)

Given a join in a graph, decide whether there exists a complete packing of cuts is an NP-complete problem.

Seymour graphs

Theorem (Seymour)

If G is a bipartite graph,
then for every join there exists a complete packing of cuts.

Theorem (Seymour)

If G is a bipartite graph,
then for every join there exists a complete packing of cuts.

Theorem (Seymour)

If G is a series-parallel graph,
then for every join there exists a complete packing of cuts.

Seymour graphs

Theorem (Seymour)

If G is a bipartite graph,
then for every join there exists a complete packing of cuts.

Theorem (Seymour)

If G is a series-parallel graph,
then for every join there exists a complete packing of cuts.

Definition

G is a Seymour graph
if for every join there exists a complete packing of cuts.

Seymour graphs

Theorem (Seymour)

If G is a bipartite graph, ($\Longleftrightarrow$ no odd cycle)
then for every join there exists a complete packing of cuts.

Theorem (Seymour)

If G is a series-parallel graph,
then for every join there exists a complete packing of cuts.

Definition

G is a Seymour graph
if for every join there exists a complete packing of cuts.

Seymour graphs

Theorem (Seymour)

If G is a bipartite graph, ($\Longleftrightarrow$ no odd cycle)
then for every join there exists a complete packing of cuts.

Theorem (Seymour)

If G is a series-parallel graph, ($\Longleftrightarrow$ no subdivision of K_4)
then for every join there exists a complete packing of cuts.

Definition

G is a Seymour graph
if for every join there exists a complete packing of cuts.

Seymour graphs

Theorem (Seymour)

If G is a bipartite graph, ($\Longleftrightarrow$ no odd cycle)
then for every join there exists a complete packing of cuts.

Theorem (Seymour)

If G is a series-parallel graph, ($\Longleftrightarrow$ no subdivision of K_4)
then for every join there exists a complete packing of cuts.

Definition

G is a Seymour graph $\Longleftrightarrow$?
if for every join there exists a complete packing of cuts.

Around Seymour graphs

subclasses

1. Seymour : Graphs without odd cycle,
2. Seymour : Graphs without subdivision of K_4,
3. Gerards : Graphs without odd K_4 and without odd prism,
4. Szigeti : Graphs without non-Seymour odd K_4 and without non-Seymour odd prism.

Around Seymour graphs

subclasses

1. Seymour : Graphs without odd cycle,
2. Seymour : Graphs without subdivision of K_4,
3. Gerards : Graphs without odd K_4 and without odd prism,
4. Szigeti : Graphs without non-Seymour odd K_4 and without non-Seymour odd prism.

Around Seymour graphs

subclasses

1. Seymour : Graphs without odd cycle,
2. Seymour : Graphs without subdivision of K_4,
3. Gerards : Graphs without odd K_4 and without odd prism,
4. Szigeti : Graphs without non-Seymour odd K_4 and without non-Seymour odd prism.

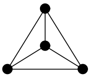
Around Seymour graphs

subclasses

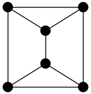
1. Seymour : Graphs without odd cycle,
2. Seymour : Graphs without subdivision of K_4,
3. Gerards : Graphs without odd K_4 and without odd prism,
4. Szigeti : Graphs without non-Seymour odd K_4 and without non-Seymour odd prism.

subclasses

1. Seymour : Graphs without odd cycle,
2. Seymour : Graphs without subdivision of K_4,
3. Gerards : Graphs without odd K_4 and without odd prism,
4. Szigeti : Graphs without non-Seymour odd K_4 and without non-Seymour odd prism.



K_4

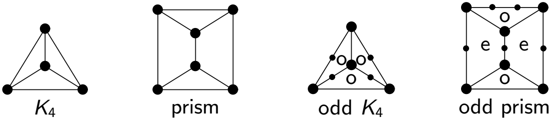


prism

Around Seymour graphs

subclasses

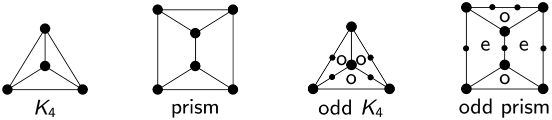
1. Seymour : Graphs without odd cycle,
2. Seymour : Graphs without subdivision of K_4,
3. Gerards : Graphs without odd K_4 and without odd prism,
4. Szigeti : Graphs without non-Seymour odd K_4 and without non-Seymour odd prism.



Around Seymour graphs

subclasses

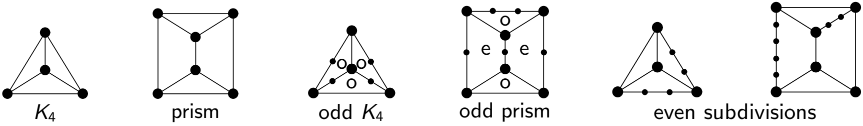
1. Seymour : Graphs without odd cycle,
2. Seymour : Graphs without subdivision of K_4,
3. Gerards : Graphs without odd K_4 and without odd prism,
4. Szigeti : Graphs without non-Seymour odd K_4 and without non-Seymour odd prism.



Around Seymour graphs

subclasses

1. Seymour : Graphs without odd cycle,
2. Seymour : Graphs without subdivision of K_4,
3. Gerards : Graphs without odd K_4 and without odd prism,
4. Szigeti : Graphs without non-Seymour odd K_4 and without non-Seymour odd prism.



Superclass

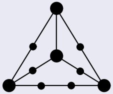
Seymour graph $\Longrightarrow$ no even subdivision of K_4 and of prism.

Attention !

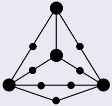
Seymour property is not inherited to subgraphs !

Attention !

Seymour property is not inherited to subgraphs !



non-Seymour
odd K_4



Seymour
graph

Preliminaries

Definition

Given a join F, a cycle C is F-tight if $|C \cap F| = |C \setminus F|$.

Remarks

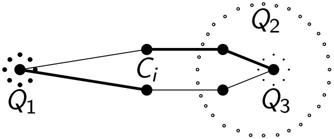
Given a join F, an F-complete packing of cuts $\mathcal{Q}$, two F-tight cycles C_1 and C_2 and a cycle C in $C_1 \cup C_2$, then

- each edge of C_i (and hence of C) belongs to a cut $Q \in \mathcal{Q}$,
- $\{C \cap Q : Q \in \mathcal{Q}, C \cap Q \neq \emptyset\}$ partitions C and $|C \cap Q|$ is even,
- $|C|$ is even so $C_1 \cup C_2$ is bipartite.

Remarks

Given a join F, an F-complete packing of cuts $\mathcal{Q}$, two F-tight cycles C_1 and C_2 and a cycle C in $C_1 \cup C_2$, then

- each edge of C_i (and hence of C) belongs to a cut $Q \in \mathcal{Q}$,
- $\{C \cap Q : Q \in \mathcal{Q}, C \cap Q \neq \emptyset\}$ partitions C and $|C \cap Q|$ is even,
- $|C|$ is even so $C_1 \cup C_2$ is bipartite.



Preliminaries

Remarks

Given a join F, an F-complete packing of cuts $\mathcal{Q}$, two F-tight cycles C_1 and C_2 and a cycle C in $C_1 \cup C_2$, then

- each edge of C_i (and hence of C) belongs to a cut $Q \in \mathcal{Q}$,
- $\{C \cap Q : Q \in \mathcal{Q}, C \cap Q \neq \emptyset\}$ partitions C and $|C \cap Q|$ is even,
- $|C|$ is even so $C_1 \cup C_2$ is bipartite.

Preliminaries

Remarks

Given a join F, an F-complete packing of cuts $\mathcal{Q}$, two F-tight cycles C_1 and C_2 and a cycle C in $C_1 \cup C_2$, then

- each edge of C_i (and hence of C) belongs to a cut $Q \in \mathcal{Q}$,
- $\{C \cap Q : Q \in \mathcal{Q}, C \cap Q \neq \emptyset\}$ partitions C and $|C \cap Q|$ is even,
- $|C|$ is even so $C_1 \cup C_2$ is bipartite.

Preliminaries

Remarks

Given a join F, an F-complete packing of cuts $\mathcal{Q}$, two F-tight cycles C_1 and C_2 and a cycle C in $C_1 \cup C_2$, then

- each edge of C_i (and hence of C) belongs to a cut $Q \in \mathcal{Q}$,
- $\{C \cap Q : Q \in \mathcal{Q}, C \cap Q \neq \emptyset\}$ partitions C and $|C \cap Q|$ is even,
- $|C|$ is even so $C_1 \cup C_2$ is bipartite.

Lemma (Sebő)

If for a join F of G there exist two F-tight cycles whose union is not bipartite, then G is not Seymour.

Co-NP characterization of Seymour graphs

Theorem (Ageev, Kostochka, Szigeti)

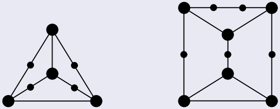
G is not Seymour if and only if G admits a join F and two F-tight cycles whose union is an odd K_4 or an odd prism.

Co-NP characterization of Seymour graphs

Theorem (Ageev, Kostochka, Szigeti)

G is not Seymour if and only if G admits a join F and two F-tight cycles whose union is an odd K_4 or an odd prism.

Examples



Seymour
odd K_4

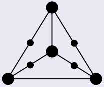
non-Seymour
odd prism

Co-NP characterization of Seymour graphs

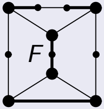
Theorem (Ageev, Kostochka, Szigeti)

G is not Seymour if and only if G admits a join F and two F-tight cycles whose union is an odd K_4 or an odd prism.

Examples



Seymour
odd K_4



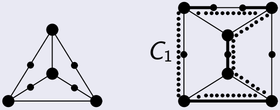
non-Seymour
odd prism

Co-NP characterization of Seymour graphs

Theorem (Ageev, Kostochka, Szigeti)

G is not Seymour if and only if G admits a join F and two F-tight cycles whose union is an odd K_4 or an odd prism.

Examples



Seymour odd K_4

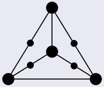
non-Seymour odd prism

Co-NP characterization of Seymour graphs

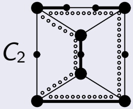
Theorem (Ageev, Kostochka, Szigeti)

G is not Seymour if and only if G admits a join F and two F-tight cycles whose union is an odd K_4 or an odd prism.

Examples



Seymour
odd K_4



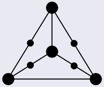
non-Seymour
odd prism

Co-NP characterization of Seymour graphs

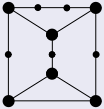
Theorem (Ageev, Kostochka, Szigeti)

G is not Seymour if and only if G admits a join F and two F-tight cycles whose union is an odd K_4 or an odd prism.

Examples



Seymour odd K_4



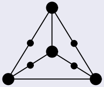
non-Seymour odd prism

Co-NP characterization of Seymour graphs

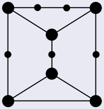
Theorem (Ageev, Kostochka, Szigeti)

G is not Seymour if and only if G admits a join F and two F-tight cycles whose union is an odd K_4 or an odd prism.

Examples



Seymour
odd K_4



non-Seymour
odd prism

Important remark

If a graph G contains as a subgraph an even subdivision of K_4 or of prism then G is not Seymour.

Forbidden minors ?

Attention !

Contraction of an edge does not keep Seymour property.

A new notion of contraction

Definitions

1. G is factor-critical if $\forall v \in V$, $G - v$ admits a perfect matching.
2. The contraction of a factor-critical subgraph and its neighbors is a factor-contraction.

A new notion of contraction

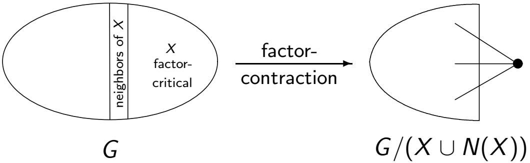
Definitions

1. G is factor-critical if $\forall v \in V$, $G - v$ admits a perfect matching.
2. The contraction of a factor-critical subgraph and its neighbors is a factor-contraction.

A new notion of contraction

Definitions

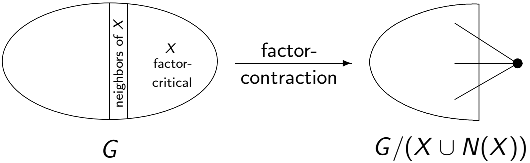
1. G is factor-critical if $\forall v \in V$, $G - v$ admits a perfect matching.
2. The contraction of a factor-critical subgraph and its neighbors is a factor-contraction.



A new notion of contraction

Definitions

1. G is factor-critical if $\forall v \in V$, $G - v$ admits a perfect matching.
2. The contraction of a factor-critical subgraph and its neighbors is a factor-contraction.



Important lemma

Factor-contraction keeps the Seymour property !

New co-NP characterization of Seymour graphs

Theorem (Ageev, Sebő, Szigeti)

G is not Seymour if and only if

- G can be factor-contracted to a graph
- that contains as a subgraph an even subdivision of K_4 or of the prism.

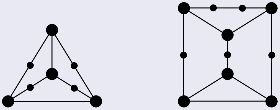
New co-NP characterization of Seymour graphs

Theorem (Ageev, Sebő, Szigeti)

G is not Seymour if and only if

- G can be factor-contracted to a graph
- that contains as a subgraph an even subdivision of K_4 or of the prism.

Examples



Seymour odd K_4 non-Seymour odd prism

Proof of sufficiency :

1. Factor-contraction keeps the Seymour property,
2. If the contracted graph H contains as a subgraph an even subdivision of K_4 or of prism then H is not Seymour.

Proof of sufficiency :

1. Factor-contraction keeps the Seymour property,
2. If the contracted graph H contains as a subgraph an even subdivision of K_4 or of prism then H is not Seymour.

Proof of sufficiency :

1. Factor-contraction keeps the Seymour property,
2. If the contracted graph H contains as a subgraph an even subdivision of K_4 or of prism then H is not Seymour.

Complete 2-packing of cuts

Complete 2-packing of cuts (for G and $F \subseteq E(G)$)

1. $2|F|$ cuts so that
2. every edge of G belongs to ≤ 2 cuts and
3. every cut contains exactly one edge of F.

Complete 2-packing of cuts

Complete 2-packing of cuts (for G and $F \subseteq E(G)$)

1. $2|F|$ cuts so that
2. every edge of G belongs to ≤ 2 cuts and
3. every cut contains exactly one edge of F.

Complete 2-packing of cuts

Complete 2-packing of cuts (for G and $F \subseteq E(G)$)

1. $2|F|$ cuts so that
2. every edge of G belongs to ≤ 2 cuts and
3. every cut contains exactly one edge of F.

Complete 2-packing of cuts

Complete 2-packing of cuts (for G and $F \subseteq E(G)$)

1. $2|F|$ cuts so that
2. every edge of G belongs to ≤ 2 cuts and
3. every cut contains exactly one edge of F.

Complete 2-packing of cuts

Complete 2-packing of cuts (for G and $F \subseteq E(G)$)

1. $2|F|$ cuts so that
2. every edge of G belongs to ≤ 2 cuts and
3. every cut contains exactly one edge of F.

Example : If $\mathcal{Q}$ is a CPC, then $2\mathcal{Q}$ is a C2PC.

Complete 2-packing of cuts

Complete 2-packing of cuts (for G and $F \subseteq E(G)$)

1. $2|F|$ cuts so that
2. every edge of G belongs to ≤ 2 cuts and
3. every cut contains exactly one edge of F.

Theorem (Edmonds-Johnson, Lovász)

F is a join $\iff$ there exists a complete 2-packing of cuts.

Complete 2-packing of cuts

Complete 2-packing of cuts (for G and $F \subseteq E(G)$)

1. $2|F|$ cuts so that
2. every edge of G belongs to ≤ 2 cuts and
3. every cut contains exactly one edge of F.

Theorem (Sebő)

Let G be a graph, $F \neq \emptyset$ a join, $v \in V(F)$.

(a) $\exists$ an F-complete 2-packing of cuts $\{\delta(X) : X \in \mathcal{C}\}$ and $C \in \mathcal{C}$ st

1. $G[C]$ is factor-critical,
2. $\{c\} \in \mathcal{C}$ $\forall c \in C$ (if $|C| = 1$, then C is contained twice in $\mathcal{C}$),
3. $v \notin C$. ($C \subseteq V(F) - v$.)

(b) If there exists an F-complete packing of cuts then there is one containing a star different of $\delta(v)$.

Complete 2-packing of cuts

Complete 2-packing of cuts (for G and $F \subseteq E(G)$)

1. $2|F|$ cuts so that
2. every edge of G belongs to ≤ 2 cuts and
3. every cut contains exactly one edge of F.

Theorem (Sebő)

Let G be a graph, $F \neq \emptyset$ a join, $v \in V(F)$.

(a) $\exists$ an F-complete 2-packing of cuts $\{\delta(X) : X \in \mathcal{C}\}$ and $C \in \mathcal{C}$ st

   1. $G[C]$ is factor-critical,
   2. $\{c\} \in \mathcal{C}$ $\forall c \in C$ (if $|C| = 1$, then C is contained twice in $\mathcal{C}$),
   3. $v \notin C$. ($C \subseteq V(F) - v$.)

(b) If there exists an F-complete packing of cuts then there is one containing a star different of $\delta(v)$.

Complete 2-packing of cuts

Complete 2-packing of cuts (for G and $F \subseteq E(G)$)

1. $2|F|$ cuts so that
2. every edge of G belongs to ≤ 2 cuts and
3. every cut contains exactly one edge of F.

Theorem (Sebő)

Let G be a graph, $F \neq \emptyset$ a join, $v \in V(F)$.

(a) $\exists$ an F-complete 2-packing of cuts $\{\delta(X) : X \in \mathcal{C}\}$ and $C \in \mathcal{C}$ st

1. $G[C]$ is factor-critical,
2. $\{c\} \in \mathcal{C}$ $\forall c \in C$ (if $|C| = 1$, then C is contained twice in $\mathcal{C}$),
3. $v \notin C$. ($C \subseteq V(F) - v$.)

(b) If there exists an F-complete packing of cuts then there is one containing a star different of $\delta(v)$.

Complete 2-packing of cuts

Complete 2-packing of cuts (for G and $F \subseteq E(G)$)

1. $2|F|$ cuts so that
2. every edge of G belongs to ≤ 2 cuts and
3. every cut contains exactly one edge of F.

Theorem (Sebő)

Let G be a graph, $F \neq \emptyset$ a join, $v \in V(F)$.

(a) $\exists$ an F-complete 2-packing of cuts $\{\delta(X) : X \in \mathcal{C}\}$ and $C \in \mathcal{C}$ st

1. $G[C]$ is factor-critical,
2. $\{c\} \in \mathcal{C}$ $\forall c \in C$ (if $|C| = 1$, then C is contained twice in $\mathcal{C}$),
3. $v \notin C$. ($C \subseteq V(F) - v$.)

(b) If there exists an F-complete packing of cuts then there is one containing a star different of $\delta(v)$.

Complete 2-packing of cuts

Complete 2-packing of cuts (for G and $F \subseteq E(G)$)

1. $2|F|$ cuts so that
2. every edge of G belongs to ≤ 2 cuts and
3. every cut contains exactly one edge of F.

Theorem (Sebő)

Let G be a graph, $F \neq \emptyset$ a join, $v \in V(F)$.

(a) $\exists$ an F-complete 2-packing of cuts $\{\delta(X) : X \in \mathcal{C}\}$ and $C \in \mathcal{C}$ st

1. $G[C]$ is factor-critical,
2. $\{c\} \in \mathcal{C}$ $\forall c \in C$ (if $|C| = 1$, then C is contained twice in $\mathcal{C}$),
3. $v \notin C$. ($C \subseteq V(F) - v$.)

(b) If there exists an F-complete packing of cuts then there is one containing a star different of $\delta(v)$.

Complete 2-packing of cuts

Complete 2-packing of cuts (for G and $F \subseteq E(G)$)

1. $2|F|$ cuts so that
2. every edge of G belongs to ≤ 2 cuts and
3. every cut contains exactly one edge of F.

Theorem (Sebő)

Let G be a graph, $F \neq \emptyset$ a join, $v \in V(F)$.

(a) $\exists$ an F-complete 2-packing of cuts $\{\delta(X) : X \in \mathcal{C}\}$ and $C \in \mathcal{C}$ st

1. $G[C]$ is factor-critical,
2. $\{c\} \in \mathcal{C}$ $\forall c \in C$ (if $|C| = 1$, then C is contained twice in $\mathcal{C}$),
3. $v \notin C$. ($C \subseteq V(F) - v$.)

(b) If there exists an F-complete packing of cuts then there is one containing a star different of $\delta(v)$.

Proof of necessity :

1. Minimal counter-example :
   1. G non-Seymour graph,
   2. any factor-contraction results in a Seymour graph,
   3. F a join without F-complete packing of cuts.
2. Application of Sebő's Theorem :
   1. No C2PC for (G, F) contains a star twice.
   2. Let $v \in V(F)$. Let $\mathcal{C}$ and $C \in \mathcal{C}$.
   3. Factor-contracting C, F_C is a join and G_C is Seymour.
   4. $\exists$ CPC $\mathcal{Q}'$ for (G_C, F_C) containing a star different of $\delta(v_C)$.
   5. $2\mathcal{Q}' \cup \{\delta(C)\} \cup \{\delta(c) : c \in C\}$ is a $C2PC$ for (G, F).
   6. By (2.1), $F_C = \emptyset$, that is $C = V(F) - v$.
3. Subgraph :
   1. $G' - v = C$ is factor-critical $\forall v \in V(F)$, that is
   2. G' is bicritical (and non-trivial).
4. Application of Lovász-Plummer's theorem :
   1. non-trivial bicritical graphs contain as a subgraph an even subdivision of K_4 or of the prism.

Proof of necessity :

1. Minimal counter-example :
   1. G non-Seymour graph,
   2. any factor-contraction results in a Seymour graph,
   3. F a join without F-complete packing of cuts.
2. Application of Sebő's Theorem :
   1. No C2PC for (G, F) contains a star twice.
   2. Let $v \in V(F)$. Let $\mathcal{C}$ and $C \in \mathcal{C}$.
   3. Factor-contracting C, F_C is a join and G_C is Seymour.
   4. $\exists$ CPC $\mathcal{Q}'$ for (G_C, F_C) containing a star different of $\delta(v_C)$.
   5. $2\mathcal{Q}' \cup \{\delta(C)\} \cup \{\delta(c) : c \in C\}$ is a $C2PC$ for (G, F).
   6. By (2.1), $F_C = \emptyset$, that is $C = V(F) - v$.
3. Subgraph :
   1. $G' - v = C$ is factor-critical $\forall v \in V(F)$, that is
   2. G' is bicritical (and non-trivial).
4. Application of Lovász-Plummer's theorem :
   1. non-trivial bicritical graphs contain as a subgraph an even subdivision of K_4 or of the prism.

Proof of necessity :

1. Minimal counter-example :
   1. G non-Seymour graph,
   2. any factor-contraction results in a Seymour graph,
   3. F a join without F-complete packing of cuts.
2. Application of Sebő's Theorem :
   1. No C2PC for (G, F) contains a star twice.
   2. Let $v \in V(F)$. Let $\mathcal{C}$ and $C \in \mathcal{C}$.
   3. Factor-contracting C, F_C is a join and G_C is Seymour.
   4. $\exists$ CPC $\mathcal{Q}'$ for (G_C, F_C) containing a star different of $\delta(v_C)$.
   5. $2\mathcal{Q}' \cup \{\delta(C)\} \cup \{\delta(c) : c \in C\}$ is a $C2PC$ for (G, F).
   6. By (2.1), $F_C = \emptyset$, that is $C = V(F) - v$.
3. Subgraph :
   1. $G' - v = C$ is factor-critical $\forall v \in V(F)$, that is
   2. G' is bicritical (and non-trivial).
4. Application of Lovász-Plummer's theorem :
   1. non-trivial bicritical graphs contain as a subgraph an even subdivision of K_4 or of the prism.

Proof of necessity :

1. Minimal counter-example :
   1. G non-Seymour graph,
   2. any factor-contraction results in a Seymour graph,
   3. F a join without F-complete packing of cuts.
2. Application of Sebő's Theorem :
   1. No C2PC for (G, F) contains a star twice.
   2. Let $v \in V(F)$. Let $\mathcal{C}$ and $C \in \mathcal{C}$.
   3. Factor-contracting C, F_C is a join and G_C is Seymour.
   4. $\exists$ CPC $\mathcal{Q}'$ for (G_C, F_C) containing a star different of $\delta(v_C)$.
   5. $2\mathcal{Q}' \cup \{\delta(C)\} \cup \{\delta(c) : c \in C\}$ is a $C2PC$ for (G, F).
   6. By (2.1), $F_C = \emptyset$, that is $C = V(F) - v$.
3. Subgraph :
   1. $G' - v = C$ is factor-critical $\forall v \in V(F)$, that is
   2. G' is bicritical (and non-trivial).
4. Application of Lovász-Plummer's theorem :
   1. non-trivial bicritical graphs contain as a subgraph an even subdivision of K_4 or of the prism.

Proof of necessity :

1. Minimal counter-example :
   1. G non-Seymour graph,
   2. any factor-contraction results in a Seymour graph,
   3. F a join without F-complete packing of cuts.
2. Application of Sebő's Theorem :
   1. No C2PC for (G, F) contains a star twice.
   2. Let $v \in V(F)$. Let $\mathcal{C}$ and $C \in \mathcal{C}$.
   3. Factor-contracting C, F_C is a join and G_C is Seymour.
   4. $\exists$ CPC $\mathcal{Q}'$ for (G_C, F_C) containing a star different of $\delta(v_C)$.
   5. $2\mathcal{Q}' \cup \{\delta(C)\} \cup \{\delta(c) : c \in C\}$ is a $C2PC$ for (G, F).
   6. By (2.1), $F_C = \emptyset$, that is $C = V(F) - v$.
3. Subgraph :
   1. $G' - v = C$ is factor-critical $\forall v \in V(F)$, that is
   2. G' is bicritical (and non-trivial).
4. Application of Lovász-Plummer's theorem :
   1. non-trivial bicritical graphs contain as a subgraph an even subdivision of K_4 or of the prism.

Proof of necessity :

1. Minimal counter-example :
   1. G non-Seymour graph,
   2. any factor-contraction results in a Seymour graph,
   3. F a join without F-complete packing of cuts.
2. Application of Sebő's Theorem :
   1. No C2PC for (G, F) contains a star twice.
   2. Let $v \in V(F)$. Let $\mathcal{C}$ and $C \in \mathcal{C}$.
   3. Factor-contracting C, F_C is a join and G_C is Seymour.
   4. $\exists$ CPC $\mathcal{Q}'$ for (G_C, F_C) containing a star different of $\delta(v_C)$.
   5. $2\mathcal{Q}' \cup \{\delta(C)\} \cup \{\delta(c) : c \in C\}$ is a $C2PC$ for (G, F).
   6. By (2.1), $F_C = \emptyset$, that is $C = V(F) - v$.
3. Subgraph :
   1. $G' - v = C$ is factor-critical $\forall v \in V(F)$, that is
   2. G' is bicritical (and non-trivial).
4. Application of Lovász-Plummer's theorem :
   1. non-trivial bicritical graphs contain as a subgraph an even subdivision of K_4 or of the prism.

Proof of necessity :

1. Minimal counter-example :
   1. G non-Seymour graph,
   2. any factor-contraction results in a Seymour graph,
   3. F a join without F-complete packing of cuts.
2. Application of Sebő's Theorem :
   1. No C2PC for (G, F) contains a star twice.
   2. Let $v \in V(F)$. Let $\mathcal{C}$ and $C \in \mathcal{C}$.
   3. Factor-contracting C, F_C is a join and G_C is Seymour.
   4. $\exists$ CPC $\mathcal{Q}'$ for (G_C, F_C) containing a star different of $\delta(v_C)$.
   5. $2\mathcal{Q}' \cup \{\delta(C)\} \cup \{\delta(c) : c \in C\}$ is a $C2PC$ for (G, F).
   6. By (2.1), $F_C = \emptyset$, that is $C = V(F) - v$.
3. Subgraph :
   1. $G' - v = C$ is factor-critical $\forall v \in V(F)$, that is
   2. G' is bicritical (and non-trivial).
4. Application of Lovász-Plummer's theorem :
   1. non-trivial bicritical graphs contain as a subgraph an even subdivision of K_4 or of the prism.

Proof of necessity :

1. Minimal counter-example :
   1. G non-Seymour graph,
   2. any factor-contraction results in a Seymour graph,
   3. F a join without F-complete packing of cuts.
2. Application of Sebő's Theorem :
   1. No C2PC for (G, F) contains a star twice.
   2. Let $v \in V(F)$. Let $\mathcal{C}$ and $C \in \mathcal{C}$.
   3. Factor-contracting C, F_C is a join and G_C is Seymour.
   4. $\exists$ CPC $\mathcal{Q}'$ for (G_C, F_C) containing a star different of $\delta(v_C)$.
   5. $2\mathcal{Q}' \cup \{\delta(C)\} \cup \{\delta(c) : c \in C\}$ is a $C2PC$ for (G, F).
   6. By (2.1), $F_C = \emptyset$, that is $C = V(F) - v$.
3. Subgraph :
   1. $G' - v = C$ is factor-critical $\forall v \in V(F)$, that is
   2. G' is bicritical (and non-trivial).
4. Application of Lovász-Plummer's theorem :
   1. non-trivial bicritical graphs contain as a subgraph an even subdivision of K_4 or of the prism.

Proof of necessity :

1. Minimal counter-example :
   1. G non-Seymour graph,
   2. any factor-contraction results in a Seymour graph,
   3. F a join without F-complete packing of cuts.
2. Application of Sebő's Theorem :
   1. No C2PC for (G, F) contains a star twice.
   2. Let $v \in V(F)$. Let $\mathcal{C}$ and $C \in \mathcal{C}$.
   3. Factor-contracting C, F_C is a join and G_C is Seymour.
   4. $\exists$ CPC $\mathcal{Q}'$ for (G_C, F_C) containing a star different of $\delta(v_C)$.
   5. $2\mathcal{Q}' \cup \{\delta(C)\} \cup \{\delta(c) : c \in C\}$ is a $C2PC$ for (G, F).
   6. By (2.1), $F_C = \emptyset$, that is $C = V(F) - v$.
3. Subgraph :
   1. $G' - v = C$ is factor-critical $\forall v \in V(F)$, that is
   2. G' is bicritical (and non-trivial).
4. Application of Lovász-Plummer's theorem :
   1. non-trivial bicritical graphs contain as a subgraph an even subdivision of K_4 or of the prism.

Proof of necessity :

1. Minimal counter-example :
   1. G non-Seymour graph,
   2. any factor-contraction results in a Seymour graph,
   3. F a join without F-complete packing of cuts.
2. Application of Sebő's Theorem :
   1. No C2PC for (G, F) contains a star twice.
   2. Let $v \in V(F)$. Let $\mathcal{C}$ and $C \in \mathcal{C}$.
   3. Factor-contracting C, F_C is a join and G_C is Seymour.
   4. $\exists$ CPC $\mathcal{Q}'$ for (G_C, F_C) containing a star different of $\delta(v_C)$.
   5. $2\mathcal{Q}' \cup \{\delta(C)\} \cup \{\delta(c) : c \in C\}$ is a $C2PC$ for (G, F).
   6. By (2.1), $F_C = \emptyset$, that is $C = V(F) - v$.
3. Subgraph :
   1. $G' - v = C$ is factor-critical $\forall v \in V(F)$, that is
   2. G' is bicritical (and non-trivial).
4. Application of Lovász-Plummer's theorem :
   1. non-trivial bicritical graphs contain as a subgraph an even subdivision of K_4 or of the prism.

Proof of necessity :

1. Minimal counter-example :
   1. G non-Seymour graph,
   2. any factor-contraction results in a Seymour graph,
   3. F a join without F-complete packing of cuts.
2. Application of Sebő's Theorem :
   1. No C2PC for (G, F) contains a star twice.
   2. Let $v \in V(F)$. Let $\mathcal{C}$ and $C \in \mathcal{C}$.
   3. Factor-contracting C, F_C is a join and G_C is Seymour.
   4. $\exists$ CPC $\mathcal{Q}'$ for (G_C, F_C) containing a star different of $\delta(v_C)$.
   5. $2\mathcal{Q}' \cup \{\delta(C)\} \cup \{\delta(c) : c \in C\}$ is a $C2PC$ for (G, F).
   6. By (2.1), $F_C = \emptyset$, that is $C = V(F) - v$.
3. Subgraph :
   1. $G' - v = C$ is factor-critical $\forall v \in V(F)$, that is
   2. G' is bicritical (and non-trivial).
4. Application of Lovász-Plummer's theorem :
   1. non-trivial bicritical graphs contain as a subgraph an even subdivision of K_4 or of the prism.

Proof of necessity :

1. Minimal counter-example :
   1. G non-Seymour graph,
   2. any factor-contraction results in a Seymour graph,
   3. F a join without F-complete packing of cuts.
2. Application of Sebő's Theorem :
   1. No C2PC for (G, F) contains a star twice.
   2. Let $v \in V(F)$. Let $\mathcal{C}$ and $C \in \mathcal{C}$.
   3. Factor-contracting C, F_C is a join and G_C is Seymour.
   4. $\exists$ CPC $\mathcal{Q}'$ for (G_C, F_C) containing a star different of $\delta(v_C)$.
   5. $2\mathcal{Q}' \cup \{\delta(C)\} \cup \{\delta(c) : c \in C\}$ is a $C2PC$ for (G, F).
   6. By (2.1), $F_C = \emptyset$, that is $C = V(F) - v$.
3. Subgraph :
   1. $G' - v = C$ is factor-critical $\forall v \in V(F)$, that is
   2. G' is bicritical (and non-trivial).
4. Application of Lovász-Plummer's theorem :
   1. non-trivial bicritical graphs contain as a subgraph an even subdivision of K_4 or of the prism.

Proof of necessity :

1. Minimal counter-example :
   1. G non-Seymour graph,
   2. any factor-contraction results in a Seymour graph,
   3. F a join without F-complete packing of cuts.
2. Application of Sebő's Theorem :
   1. No C2PC for (G, F) contains a star twice.
   2. Let $v \in V(F)$. Let $\mathcal{C}$ and $C \in \mathcal{C}$.
   3. Factor-contracting C, F_C is a join and G_C is Seymour.
   4. $\exists$ CPC $\mathcal{Q}'$ for (G_C, F_C) containing a star different of $\delta(v_C)$.
   5. $2\mathcal{Q}' \cup \{\delta(C)\} \cup \{\delta(c) : c \in C\}$ is a $C2PC$ for (G, F).
   6. By (2.1), $F_C = \emptyset$, that is $C = V(F) - v$.
3. Subgraph :
   1. $G' - v = C$ is factor-critical $\forall v \in V(F)$, that is
   2. G' is bicritical (and non-trivial).
4. Application of Lovász-Plummer's theorem :
   1. non-trivial bicritical graphs contain as a subgraph an even subdivision of K_4 or of the prism.

Proof of necessity :

1. Minimal counter-example :
   1. G non-Seymour graph,
   2. any factor-contraction results in a Seymour graph,
   3. F a join without F-complete packing of cuts.
2. Application of Sebő's Theorem :
   1. No C2PC for (G, F) contains a star twice.
   2. Let $v \in V(F)$. Let $\mathcal{C}$ and $C \in \mathcal{C}$.
   3. Factor-contracting C, F_C is a join and G_C is Seymour.
   4. $\exists$ CPC $\mathcal{Q}'$ for (G_C, F_C) containing a star different of $\delta(v_C)$.
   5. $2\mathcal{Q}' \cup \{\delta(C)\} \cup \{\delta(c) : c \in C\}$ is a $C2PC$ for (G, F).
   6. By (2.1), $F_C = \emptyset$, that is $C = V(F) - v$.
3. Subgraph : Let $G' := G[V(F)]$.
   1. $G' - v = C$ is factor-critical $\forall v \in V(F)$, that is
   2. G' is bicritical (and non-trivial).
4. Application of Lovász-Plummer's theorem :
   1. non-trivial bicritical graphs contain as a subgraph an even subdivision of K_4 or of the prism.

Proof of necessity :

1. Minimal counter-example :
   1. G non-Seymour graph,
   2. any factor-contraction results in a Seymour graph,
   3. F a join without F-complete packing of cuts.
2. Application of Sebő's Theorem :
   1. No C2PC for (G, F) contains a star twice.
   2. Let $v \in V(F)$. Let $\mathcal{C}$ and $C \in \mathcal{C}$.
   3. Factor-contracting C, F_C is a join and G_C is Seymour.
   4. $\exists$ CPC $\mathcal{Q}'$ for (G_C, F_C) containing a star different of $\delta(v_C)$.
   5. $2\mathcal{Q}' \cup \{\delta(C)\} \cup \{\delta(c) : c \in C\}$ is a $C2PC$ for (G, F).
   6. By (2.1), $F_C = \emptyset$, that is $C = V(F) - v$.
3. Subgraph : Let $G' := G[V(F)]$.
   1. $G' - v = C$ is factor-critical $\forall v \in V(F)$, that is
   2. G' is bicritical (and non-trivial).
4. Application of Lovász-Plummer's theorem :
   1. non-trivial bicritical graphs contain as a subgraph an even subdivision of K_4 or of the prism.

Proof of necessity :

1. Minimal counter-example :
   1. G non-Seymour graph,
   2. any factor-contraction results in a Seymour graph,
   3. F a join without F-complete packing of cuts.
2. Application of Sebő's Theorem :
   1. No C2PC for (G, F) contains a star twice.
   2. Let $v \in V(F)$. Let $\mathcal{C}$ and $C \in \mathcal{C}$.
   3. Factor-contracting C, F_C is a join and G_C is Seymour.
   4. $\exists$ CPC $\mathcal{Q}'$ for (G_C, F_C) containing a star different of $\delta(v_C)$.
   5. $2\mathcal{Q}' \cup \{\delta(C)\} \cup \{\delta(c) : c \in C\}$ is a $C2PC$ for (G, F).
   6. By (2.1), $F_C = \emptyset$, that is $C = V(F) - v$.
3. Subgraph : Let $G' := G[V(F)]$.
   1. $G' - v = C$ is factor-critical $\forall v \in V(F)$, that is
   2. G' is bicritical (and non-trivial).
4. Application of Lovász-Plummer's theorem :
   1. non-trivial bicritical graphs contain as a subgraph an even subdivision of K_4 or of the prism.

Proof of necessity :

1. Minimal counter-example :
   1. G non-Seymour graph,
   2. any factor-contraction results in a Seymour graph,
   3. F a join without F-complete packing of cuts.
2. Application of Sebő's Theorem :
   1. No C2PC for (G, F) contains a star twice.
   2. Let $v \in V(F)$. Let $\mathcal{C}$ and $C \in \mathcal{C}$.
   3. Factor-contracting C, F_C is a join and G_C is Seymour.
   4. $\exists$ CPC $\mathcal{Q}'$ for (G_C, F_C) containing a star different of $\delta(v_C)$.
   5. $2\mathcal{Q}' \cup \{\delta(C)\} \cup \{\delta(c) : c \in C\}$ is a $C2PC$ for (G, F).
   6. By (2.1), $F_C = \emptyset$, that is $C = V(F) - v$.
3. Subgraph : Let $G' := G[V(F)]$.
   1. $G' - v = C$ is factor-critical $\forall v \in V(F)$, that is
   2. G' is bicritical (and non-trivial).
4. Application of Lovász-Plummer's theorem :
   1. non-trivial bicritical graphs contain as a subgraph an even subdivision of K_4 or of the prism.

What we can not do

1. Given a graph G, decide whether it is a Seymour graph.
2. Given a graph G and a join F in G, decide whether there exists an F-complete packing of cuts.

Algorithmic aspects

What we can not do

1. Given a graph G, decide whether it is a Seymour graph.
2. Given a graph G and a join F in G, decide whether there exists an F-complete packing of cuts.

Algorithmic aspects

What we can not do

1. Given a graph G, decide whether it is a Seymour graph.
2. Given a graph G and a join F in G, decide whether there exists an F-complete packing of cuts.

Algorithmic aspects

What we can not do

1. Given a graph G, decide whether it is a Seymour graph.
2. Given a graph G and a join F in G, decide whether there exists an F-complete packing of cuts.

What we can do

Given a graph G and a join F in G,

1. either provide an F-complete packing of cuts
2. or show that G is not Seymour.

Algorithmic aspects

What we can not do

1. Given a graph G, decide whether it is a Seymour graph.
2. Given a graph G and a join F in G, decide whether there exists an F-complete packing of cuts.

What we can do

Given a graph G and a join F in G,

1. either provide an F-complete packing of cuts
2. or show that G is not Seymour.

Algorithmic aspects

What we can not do

1. Given a graph G, decide whether it is a Seymour graph.
2. Given a graph G and a join F in G, decide whether there exists an F-complete packing of cuts.

What we can do

Given a graph G and a join F in G,

1. either provide an F-complete packing of cuts
2. or show that G is not Seymour.

Open problem

NP characterization ?

Open problem

NP characterization ?

Find a construction for Seymour graphs !