Conflict Graphs for Combinatorial Optimization Problems

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joint work with Andreas Darmann and Joachim Schauer

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Introduction

Combinatorial Optimization Problem CO

Consider any CO with decision variables $x_j \in \{0, 1\}$, $j \in V$, and a feasible domain $x \in S$.

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Conflict Structure

Add disjunctive constraints for some pairs of variables:

$$x_i + x_j \leq 1 \quad \text{for } (i,j) \in E \subset V \times V$$

$\Longrightarrow$ at most one of the two variables i, j can be set to 1.

Representation by a **conflict graph** $G = (V, E)$

Edges in G connect conflicting variables of CO.

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Relation to Independent Set (IS)

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Intersection of S with an independent / stable set problem in G

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One main direction of research:
Identify special graph classes for the conflict graph G such that the considered CO problem

- is polynomially solvable
- permits a (fully) polynomial approximation scheme
- has a constant approximation ratio

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- Epstein et al.'08: extension to two dimensional packing of squares
- Jansen '99: A-FPTAS for special graph classes

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e.g.: perfect, bipartite, interval, *d*-inductive graphs.

- Gendreau et al.'04: heuristics and lower bounds
- Malaguti et al.'07: hybrid tabu search
- Malaguti et al.'08: exact algorithm (branch-and-price)

Conflict Graphs: Scheduling

Mutual Exclusion Scheduling

Schedule unit-length jobs on m machines, conflicting jobs not to be executed in the same time interval.

Baker, Coffman '96; Bodlaender, Jansen '93;
polynomially solvable special cases, special graph classes.

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Scheduling with Incompatible Jobs

Conflicting jobs not to be executed on the same machine.

Bodlaender, Jansen, Woeginger '94:
approximation algorithms for special graph classes.

Knapsack Problem with Conflict Graph (KCG)

Vertices (=items) adjacent in G cannot be packed together in the knapsack!

ILP-Formulation

$$
\begin{array}{lll}
\text{(KCG)} & \max & \sum_{j=1}^{n} p_j x_j \\
 & \text{s.t.} & \sum_{j=1}^{n} w_j x_j \leq c \\
 & & (i,j) \in E \implies x_i + x_j \leq 1 \\
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\end{aligned}
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Introduce upper bound P on optimal solution, e.g. $P := \sum_{j=1}^{n} p_j$

Note: Classical Greedy algorithm can perform as bad as possible!

Literature on KCG

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no special graph classes considered!

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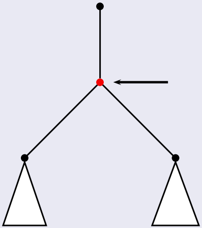
Pseudo-polynomial time algorithms and FPTAS for KCG on:

- Trees
- Graphs with Bounded Treewidth
- Chordal Graphs

Idea for KCG on Trees

Basic Observation



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$\Longrightarrow$ Scaling yields FPTAS
We use Dynamic Programming by Reaching moving bottom up in the conflict tree.

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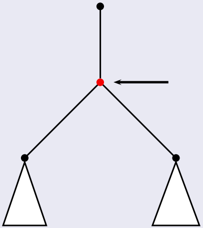

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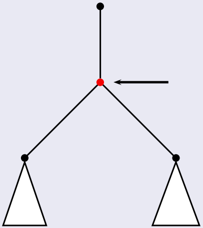

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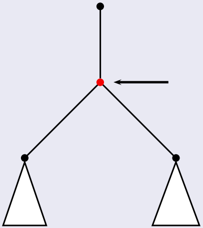

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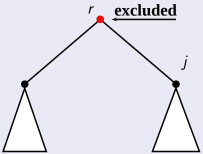
$y_i(d)$ solution with profit d and minimal weight found in the subtree $T(i)$ with item i **excluded**.

Algorithmic Details

Tree traversed in Depth-First-Search-Order (DFS)

Vertex r excluded



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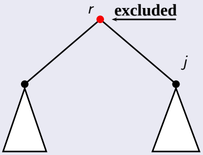

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$$y_r(d) = \min_k \{y_r(d-k) + \min\{z_j(k), y_j(k)\}\}$$

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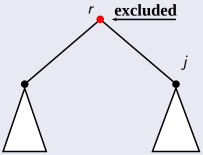

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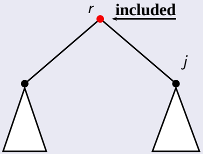

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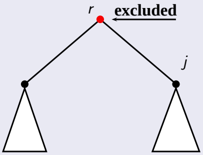

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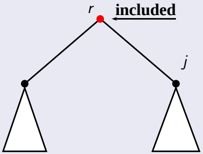

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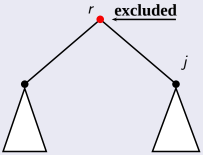

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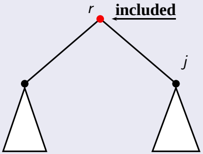

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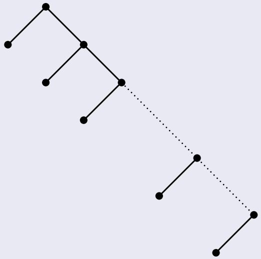
Running time: $O(nP^2)$ Space: $O(nP)$ (trivial version)

Space Reduction Method

Worst case without reduction: $O(nP)$



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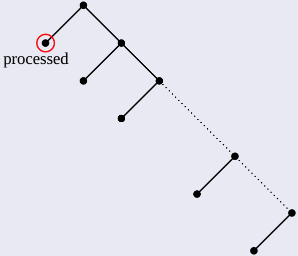

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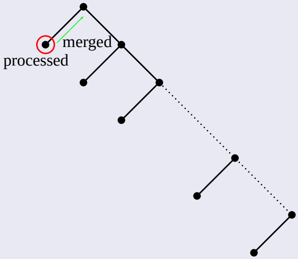

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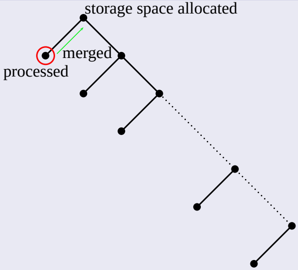

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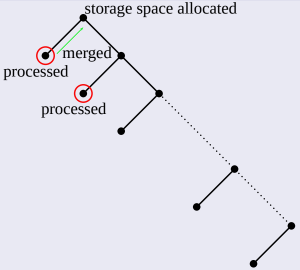

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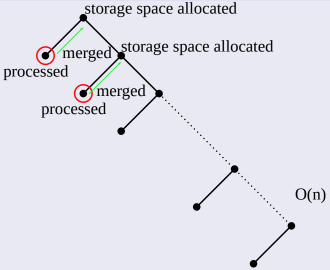

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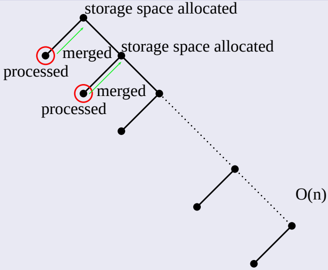

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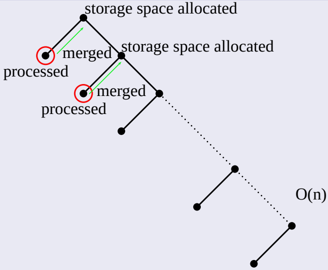

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In general: Space can be reduced to $O(\log n)$ arrays.

Worst-case: Complete binary tree

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Property 1. Tree T processed in DFS order with following rule: Take child vertex j, whose subtree contains the largest number of vertices.

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Lemma 1. An algorithm A fulfilling Properties 1, 2 and 3 uses at most $(\mathrm{ld}(n) + 1) * O(k)$ space.

Sketch of Proof. r has k childs $i_1 \dots i_k$ so that $|T(i_j)| \leq \frac{n}{2}$ and w.l.o.g $|T(i_1)| \geq |T(i_j)|$ for all $j \in \{1 \dots k\}$:
Then the processing of $T(i_1)$ is done by using at most $(\mathrm{ld}(\frac{n}{2}) + 1) * O(k) = \mathrm{ld}(n) * O(k)$ space. After merging this subtree to r this space can be deallocated, but $O(k)$ space is used at vertex r, which has to be kept until A has finished.

Bounded Treewidth

Tree-Decomposition of graph $G = (V, E)$

Every graph can be represented by a tree whose vertices are subsets of V.
Original graph can be reproduced from the tree-decomposition.
E.g. adjacent vertices of G must be jointly contained in at least one subset.

Treewidth: minimal cardinality over all tree-decompositions of the largest subset-1.

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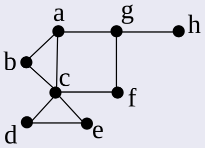

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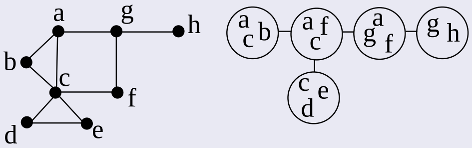

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Nice Tree-Decomposition

For algorithmic purposes, the structure of the decomposition is restricted to four simple configurations.

A nice tree-decomposition with the same treewidth can be computed from a tree-decomposition in $O(n)$ time.
[cf. Bodlaender, Koster '08]

Many $\mathcal{NP}$-hard problems can be solved efficiently for graphs with bounded treewidth. [Bodlaender '97]

Solving KCG with Bounded Treewidth

Algorithmic Idea

Take a nice tree-decomposition T of G
Process T in a DFS way.

For every vertex of T (i.e. a subset of V):
Consider all independent sets (IS) of all vertices in T explicitly.
Note: Number of IS is constant!

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Time and Space

KCG for conflict graphs of bounded treewidth can be solved in $O(nP^2)$ time and $O(\log n \; P + n)$ space given a tree-decomposition.

The space reduction method of Lemma 1 can be applied again!

Chordal Graphs

Definition

A Chordal Graph (a.k.a. triangulated graph) does not contain induced cycles other than triangles.

$\implies$ every cycle of at least four vertices has a **chord**.

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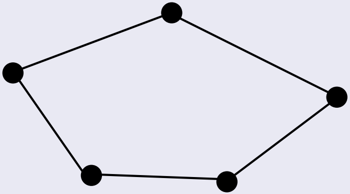

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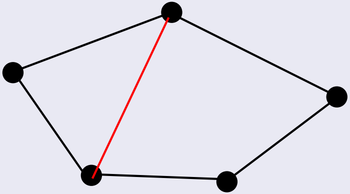

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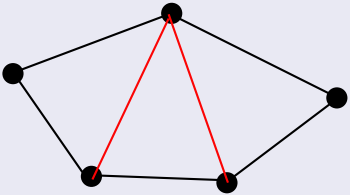
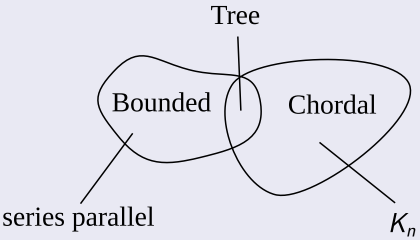


Chordal Graphs

Bounded Treewidth versus Chordal Graph

No subset relation!



Solving KCG on Chordal Graphs

Tree Representation

For every chordal graph G there is a clique tree $T = (\mathcal{K}, \mathcal{E})$:

maximal cliques K of G are vertices of T

for each vertex $v \in G$:

all cliques K containing v induce a subtree in T.

cf. [Blair, Peyton '93]

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Basic Idea of the Algorithm

The vertices of the clique tree T are complete subgraphs.

$\Longrightarrow$ at most one vertex of each clique can be in the knapsack.

Process T in a DFS way.

Solving KCG on Chordal Graphs

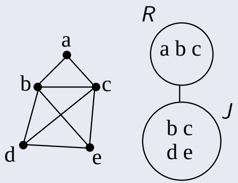
Dynamic Programming Definition

$f_d^v(I)$:
solution with profit d and minimal weight containing item $v \in I$, while considering in the clique tree only the subtree rooted in I.

definition extended to $v = \emptyset$.

Solving KCG on Chordal Graphs

Vertex R with one (or first) child J



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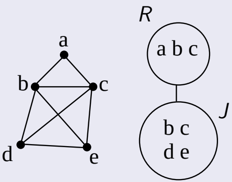
for $v \in R$:

if $v \in R \cap J$: $\{b, c\}$

$$f_d^v(R) = f_d^v(J)$$



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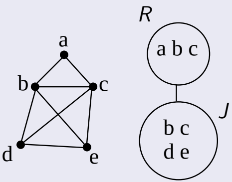

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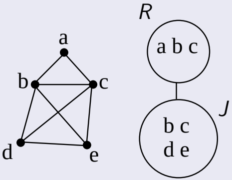
$$f_d^v(R) = f_d^v(J)$$

else: $\{a\}$

$$f_d^v(R) = w(v) + \min_i \left\{ f^i_{d-p(v)}(J) : i \in (J \setminus R) \cup \emptyset \right\}$$

Solving KCG on Chordal Graphs

Vertex R with one (or first) child J



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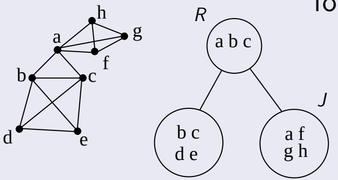
Solving KCG on Chordal Graphs

Vertex R with second child J



for $v \in R$:

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$$f_d^v(R) =$$
$$\min_k \left\{ f_k^v(R) + f_{d-k+p(v)}^v(J) \right\}$$
$$f_d^v(R) = f_d^v(R) - w(v)$$

Solving KCG on Chordal Graphs

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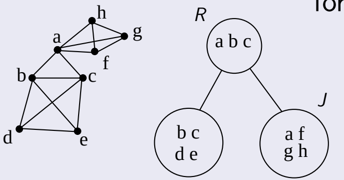

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Solving KCG on Chordal Graphs

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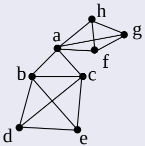

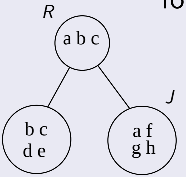

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Straightforward: $O(n^4 P^2)$

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$O(n \log n \; P)$ with space reduction technique!

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Storing Solution Sets (for all three algorithms)

Note: Storing not only solution values but solution sets increases time and space by a factor of n (or $\log n$ for bit-encoding).

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This can be avoided by applying a general recursive divide and conquer technique (see Pferschy '99).

Deriving an FPTAS from Dynamic Programming

Scaling

Classical approach of an FPTAS for the knapsack problem:

Scale profits:

$$\tilde{p}_j = \left(\frac{n}{\varepsilon p_{\max}}\right) p_j$$

Running time:

$$n \cdot P \quad \xrightarrow{\text{scaling}} n \cdot \tilde{P} \leq n \cdot n \, \tilde{p}_{\max} = n^2 \left(\frac{n}{\varepsilon p_{\max}}\right) p_{\max} = \frac{n^3}{\varepsilon}$$

Induced relative error can be bounded by ε.

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FPTAS for KCG

Same scaling approach can be applied to all three KCG algorithms.
Technical details rather straightforward.

Minimum Spanning Tree with Conflict Graph (MSTCG)

Problem Description

given:

a weighted graph $H = (V, E, w)$,

a conflict graph $G = (E, \bar{E})$ whose m vertices correspond uniquely to edges in E.

edge $\bar{e} = (i, j) \in \bar{E} \implies$ conflict between the two edges $i, j \in E$

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Problem (MSTCG):

find a minimum spanning tree $T \subseteq E$ of H
without conflicts w.r.t. G

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Question:
For which type of conflict graph does MSTCG become $\mathcal{NP}$-hard?

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2-**ladder**: graph whose components are paths of length one.

3-**ladder**: graph whose components are paths of length two.

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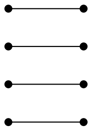
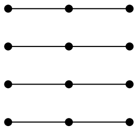
2-ladder



3-ladder



(imagine a rope ladder)

Minimum Spanning Tree with Conflict Graph (MSTCG)

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For which type of conflict graph does MSTCG become $\mathcal{NP}$-hard?

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2-**ladder**: graph whose components are paths of length one.

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Our Complexity Results

- MSTCG is polynomially solvable if the conflict graph G is a 2-ladder.

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Our Complexity Results

- MSTCG is polynomially solvable if the conflict graph G is a 2-ladder.
- MSTCG is strongly $\mathcal{NP}$-hard if the conflict graph G is a 3-ladder.

MSTCG with a 2-ladder

Conflict-free Matroid

Subsets of edges NOT containing any conflicting pair define the **conflict-free matroid**:

$$\mathcal{I} := \left\{ E' \subseteq E \mid \nexists (e,f) \in \bar{E} : \{e,f\} \subseteq E' \right\}$$

Note that in a 2-ladder conflicting pairs are independent from each other.

MSTCG with a 2-ladder

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Note that in a 2-ladder conflicting pairs are independent from each other.

$\Longrightarrow$ MSTCG is the intersection of the *graphic matroid* (MST) with the *conflict-free matroid*

Matroid Intersection

MSTCG with a 2-ladder can be solved by Edmonds' weighted matroid intersection algorithm in polynomial time.

MSTCG with a 3-ladder

Complexity Result

MSTCG with a 3-ladder is strongly $\mathcal{NP}$-hard.

MSTCG with a 3-ladder

Complexity Result

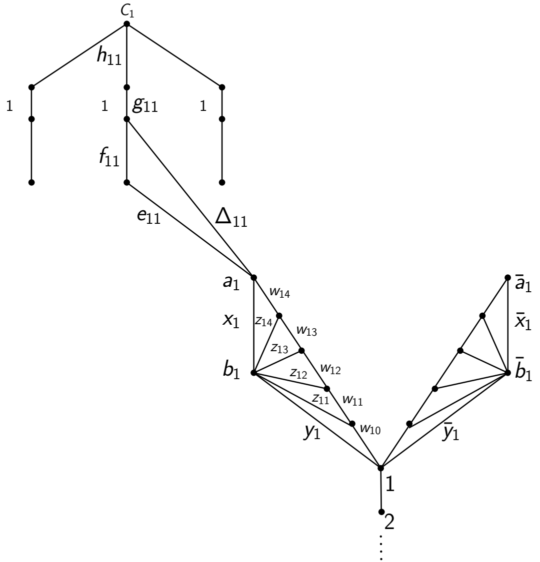
MSTCG with a 3-ladder is strongly $\mathcal{NP}$-hard.

Construction

Reduction from the special case of 3-SAT, where each variable occurs in at most 5 clauses (still $\mathcal{NP}$-complete).

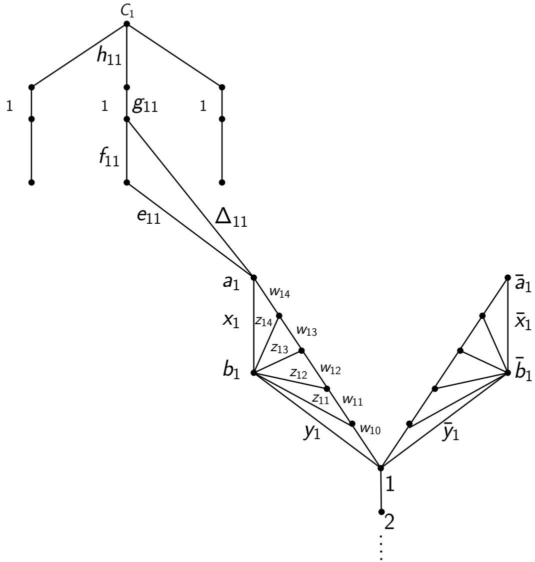
Construct an instance of MST and a 3-ladder conflict graph as follows:

MSTCG with a 3-ladder



For every clause, e.g. C_1, a *fork* represents the three literals.

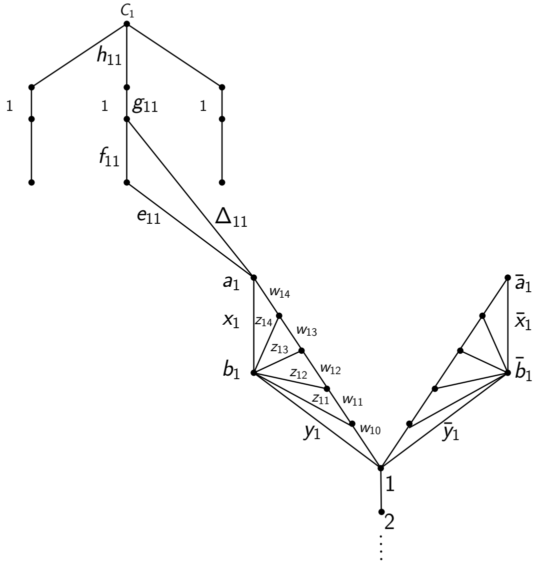
MSTCG with a 3-ladder



For every clause, e.g. C_1, a *fork* represents the three literals.

For every variable, two *gadgets* represent x_1 and $\bar{x}_1$.
Edges w_{1j}, z_{1j} for the ≤ 5 clauses variable 1 appears in.

MSTCG with a 3-ladder



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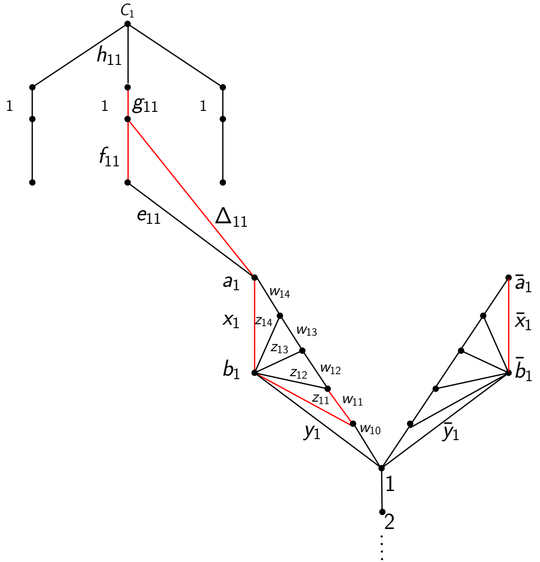
For every variable, two *gadgets* represent x_1 and $\bar{x}_1$.
Edges w_{1j}, z_{1j} for the ≤ 5 clauses variable 1 appears in.

Spanning tree:
Connection from C_1 to 1 either via x_1 or via $w_{10}, \ldots, w_{14}$.

edge x_1 in the tree $\Leftrightarrow$ $x_1 = \text{TRUE}$

MSTCG with a 3-ladder



Conflict Graph G:

$(x_1, \bar{x}_1)$

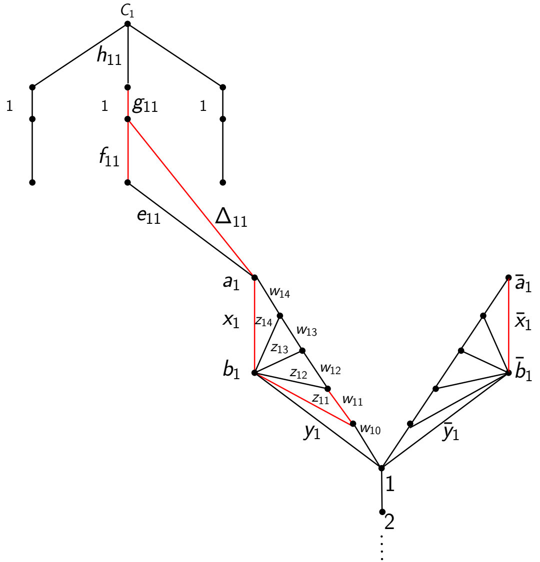
(Δ_{11}, g_{11})

(z_{11}, w_{11}, f_{11})

(z_{1j}, w_{1j}, f_{1j}), $j = 2, 3, 4$

(w_{10}, f_{10})

MSTCG with a 3-ladder



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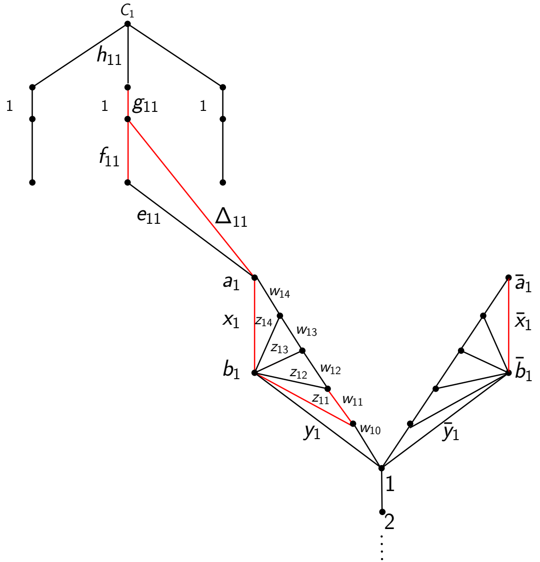
Connect C_1 to 1 in MST:
choose $g_{11} \longrightarrow f_{11} \Rightarrow$
w_{11} forbidden
other w_{1j}: either forbidden or chosen $\Rightarrow$
z_{1j} forbidden $\Longrightarrow$
only connection via x_1

MSTCG with a 3-ladder



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MST: exactly one edge g_{ij} for every clause j

MSTCG with a 3-ladder

Construction

It can be shown:
There is a truth assignment for a 3-Sat instance with k clauses
$\Longleftrightarrow$
there is a conflict-free spanning tree with weight $\leq k$.

MSTCG with a 3-ladder

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Conclusion

Conflicts makes optimization (and life in general) much more difficult.

$\Longrightarrow$ try to avoid conflicts whenever possible!

Thank you for your attention!