A Betweenness Approach for Solving the Linear Arrangement Problem

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Joint work with
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Aussois, January 16th, 2009

Outline

Definition

- Given an undirected graph $G(V, E)$

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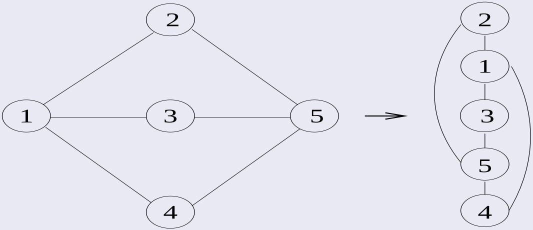
Example

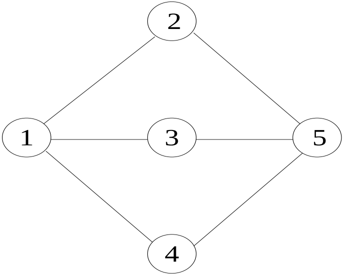


We call the minimum $lap(G)$. Here $lap(G) = 10$.

Combinatorial Lower Bounds I

Degree Lower Bound, Petit (2003)

$$\mathsf{LB_D} = \frac{1}{2}\sum_{i \in V} \lfloor (\deg(i)+1)^2/4 \rfloor$$

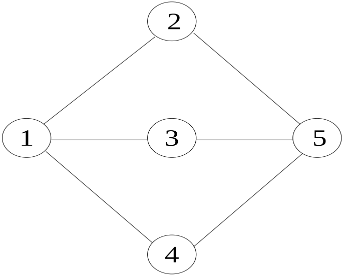


$\mathsf{LB_D} = \frac{1}{2}(4+4+2+2+2) = 7$

Combinatorial Lower Bounds II

Edge Lower Bound, Petit (2003)

At most $(n-1)$ edges with distance 1, $(n-2)$ with distance 2, etc.



$\mathsf{LB_E} = 4 \times 1 + 2 \times 2 = 8$

Eigenvalue Lower Bound

Eigenvalue Lower Bound, Juvan, Mohar (1992)

The $n \times n$ Laplacian matrix $L(G)$ of G is defined as:

$$L(G)_{i,j} := \begin{cases} \deg(i) & i = j \\ -1 & (i,j) \in E \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathrm{LB}_{\mathrm{JM}} = \lceil (\lambda_2 (n^2 - 1)/6 \rceil$$

$$L(G) = \begin{pmatrix} 3 & -1 & -1 & -1 & 0 \\ -1 & 2 & 0 & 0 & -1 \\ -1 & 0 & 2 & 0 & -1 \\ -1 & 0 & 0 & 2 & -1 \\ 0 & -1 & -1 & -1 & 3 \end{pmatrix}$$

$$\lambda_2 = 2 \quad \Rightarrow \quad \mathrm{LB}_{\mathrm{JM}} = \lceil (2(5^2 - 1)/6 \rceil = 8$$

Comparison of the Bounds

Name	n	m	UB	$\mathrm{LB}_{\mathrm{JM}}$	LB_{E}	LB_{D}
gd95c	62	144	506	37	250	292
gd96b	111	193	1416	42	276	702
gd96c	65	125	519	37	186	191
gd96d	180	228	2391	418	277	595

Linear Programming Bound

Amaral, Caprara, Letchford, Salazar (2007)

Introduce distance variables d_{ij} and solve:

$$\min \sum_{(i,j)\in E} d_{ij}$$

$$s.t. \sum_{(i,j)\in E(G')} d_{ij} \geq lap(G'), \quad G' \text{ subgraph of } G$$

$$d_{ij} \geq 1, \quad (i,j) \in E,$$

where $lap(G')$ is known, for example for G' stars, cliques, etc.

Note: Separation can be done in the complete graph after computing shortest paths on the d_{ij}-values.

Betweenness Approach

Indicators

For each triple i, k, j, $(i, j) \in E, k \in V \setminus \{i, j\}$ and each order π χ^{π}_{ikj} indicates whether k lies between i and j in π:

$$\chi^{\pi}_{ikj} := \begin{cases} 1 & \pi^{-1}(i) < \pi^{-1}(k) < \pi^{-1}(j) \quad \text{or} \\ & \pi^{-1}(i) > \pi^{-1}(k) > \pi^{-1}(j) \\ 0 & \text{otherwise.} \end{cases}$$

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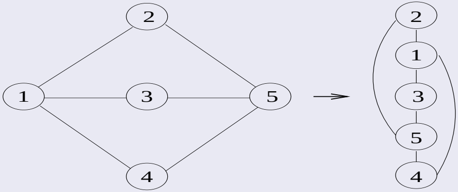

Here: $\chi^{\pi}_{132} = 0$ and $\chi^{\pi}_{235} = 1$

Reformulation of the Problem

Computing the distances

$$|\pi^{-1}(i) - \pi^{-1}(j)| = 1 + \sum_{k} \chi^{\pi}_{ikj}$$

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Linear Arrangement Problem

$$\min_{\pi \in S(n)} \sum_{(i,j) \in E} (1 + \sum_{k} \chi^{\pi}_{ikj}) = m + \min_{\pi \in S(n)} \sum_{(i,j) \in E} \sum_{k \in V} \chi^{\pi}_{ikj}.$$

Branch-and-Cut based on Consecutive Ones

Problem

Given a $0/1$ vector x. Is there a labeling π with $x = \chi^\pi$?

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Writing x as matrix

We define a matrix $M(x) \in \{0,1\}^{2m \times n}$ like follows:

$$M(x)_{r,k} := \begin{cases} 1 & k = i \text{ and } r = r(i,j) \text{ or} \\ & k = j \text{ and } r = r(i,j) + m \\ 0 & k = j \text{ and } r = r(i,j) \text{ or} \\ & k = i \text{ and } r = r(i,j) + m \\ x_{ikj} & \text{otherwise,} \end{cases}$$

where $r(i,j)$ denotes the edge with endnodes i and j.

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where $r(i,j)$ denotes the edge with endnodes i and j.

Observation

x is a feasible betweenness vector if and only if $M(x)$ has the *consecutive ones property for rows*.

Transformation into a Consecutive Ones Problem

Characterization of Tucker (1972)

A 0/1 matrix M has the consecutive ones property for rows iff none of five types of forbidden matrices occur in M as submatrix.

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There is a set of valid inequalities that can be separated in polynomial time and cut off all forbidden matrices.

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By solving the *Weighted Consecutive Ones Problem* with weights

$$w_{r,k} := \begin{cases} n & k = i \text{ and } r = r(i,j) \text{ or} \\ & k = j \text{ and } r = r(i,j) \\ -1 & \text{otherwise,} \end{cases}$$

we can solve the Linear Arrangement Problem.

Branch-and-Cut based on Betweenness-Variables

Betweenness Polytope

$$P^G_{BTW} = \mathrm{conv}\{\chi^\pi \mid \pi \in S(n)\}$$

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$$m + \min_{x \in P^G_{BTW}} \sum_{k \in V} \sum_{(i,j) \in E} x_{ikj}$$

Branch-and-Cut based on Betweenness-Variables

Betweenness Polytope

$$P_{BTW}^{G} = \mathrm{conv}\{\chi^{\pi} \mid \pi \in S(n)\}$$

Linear Arrangement Problem

$$m + \min_{x \in P_{BTW}^{G}} \sum_{k \in V} \sum_{(i,j) \in E} x_{ikj}$$

Observations

- Like in the d_{ij}-formulation: Let $G' \subset G$. Each valid inequality for $P_{BTW}^{G'}$ is valid for P_{BTW}^{G}.
- For special graphs (stars, cliques, cycles, ...) we know inequalities derived from the d_{ij}-formulation.

Splitting distances

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3-Star

$$d_{12} + d_{13} + d_{14} \geq 4$$

$$\downarrow$$

$$x_{132} + x_{142} + x_{123} + x_{143} + x_{124} + x_{134} \geq 1$$

More Valid Inequalities

Observation

Not every valid inequality for P^G_{BTW} can be derived from the d_{ij}-formulation.

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Triangle Equation

For ij, ik and $jk \in E$ the triangle equation holds:

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Not every valid inequality for P_{BTW}^{G} can be derived from the d_{ij}-formulation.

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For ij, ik and $jk \in E$ the triangle equation holds:

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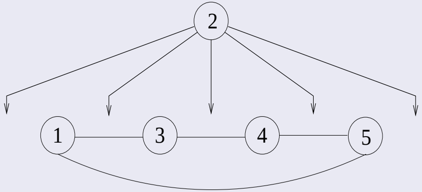
Note

The triangle equations could be easily separated but they only exist if the corresponding triangle is part of G.

Relation to the Cut-Polytope I

Observation

The vector χ^{π}_{ikj} for fixed k is incidence vector of a cut in $G \setminus \{k\}$:



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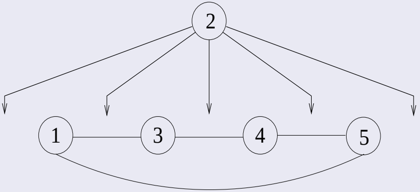

Consequence

The projection of P^G_{BTW} onto the variables x_{ikj} for a fixed node k is isomorphic to the cut polytope $P^{G'}_{CUT}$ with $G' = G \setminus \{k\}$.

Relation to the Cut-Polytope II

Jünger, Reinelt, Rinaldi (1998)

Consider an LP-solution over the semimetric polytope of a connected graph $G(V, E)$. For each missing edge $e = uv \notin E$ lower and upper bounds of the (artificial) LP value $\bar{x}_e$ are given by

$$\xi_l = \max\left\{\bar{x}(F) - \bar{x}(P \setminus F) - |F| + 1 \mid P\ (u, v)\text{-path},\ F \subseteq P,\ |F| \text{ odd}\right\}$$

$$\xi_u = \min\left\{-\bar{x}(F) + \bar{x}(P \setminus F) + |F| \mid P\ (u, v)\text{-path},\ F \subseteq P,\ |F| \text{ even}\right\}$$

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Consequence

All separation procedures for the betweenness polytope can be extended to the complete graph by shortest path computations.

Complete Description of P_{BTW}^{G} for $G = K_4$

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Triangle equations and odd-cycle inequalities are sufficient to describe the betweenness polytope of K_4.

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3-Star Inequality

$$x_{ikj} + x_{ilj} + x_{ijk} + x_{ilk} + x_{ijl} + x_{ikl} \geq 1$$

can be written as the sum of the triangle equation

$$x_{jlk} + x_{kjl} + x_{jkl} = 1$$

and 3 odd-cycle inequalities

$$x_{ilk} + x_{ilj} - x_{jlk} \geq 0$$

$$x_{ijk} + x_{ijl} - x_{kjl} \geq 0$$

$$x_{ikl} + x_{ikj} - x_{jkl} \geq 0$$

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Feasibility Test

Instead of an IP-formulation we use the PQ-Tree-Algorithm (Booth, Lueker, 1976) as feasibility test. If an integer solution is not feasible we derive a weak but violated inequality.

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- Enumerate all 5-star inequalities

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- Compute upper bounds for all $\bar{x}_{ikj}$ as described before.
- Enumerate all triangle inequalities
- Enumerate all 5-star inequalities
- Separate bigger stars heuristically

Comparison of Root Bounds

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gd95c	62	144	506	442	489	506^*
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- Using general cut generation like target-cuts or mod-k-cuts
- Using the knowledge on separating procedures for max-cut (for example shrinking)
- Better branching rules
- Behaviour of our code on planar graphs