Sorting with Complete Networks of Stacks

Felix König Marco Lübbecke

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Stack Sorting

- (complete) digraph of $|V| = k$ nodes ($\equiv$ stacks)
- source s (the input), sink t (the output)
- permutation π of $[n]$ at s

- move numbers from stack to stack along arcs
- such that they arrive at t in correct order

- goal: minimize number of shuffles

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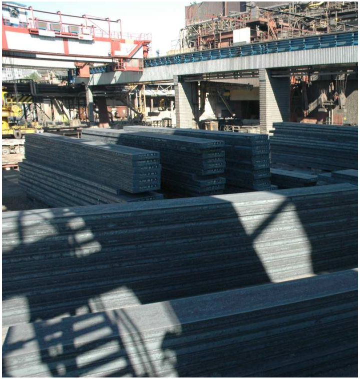
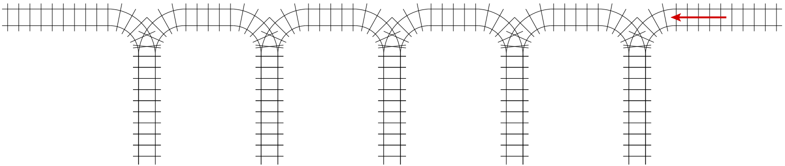
Applications



Related: Sortability of Permutations

- "railway sidings" Knuth (1968)
- rail cars arrive permuted on one end
- leave on other end in correct order
- rail cars cannot move back to previous tracks

- stack sorting problems $\leftrightarrow$ permutation patterns



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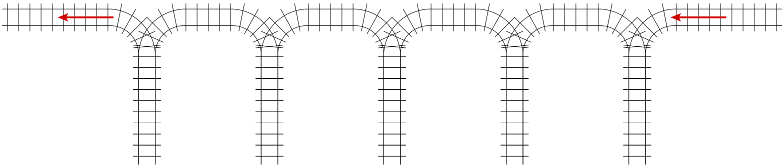

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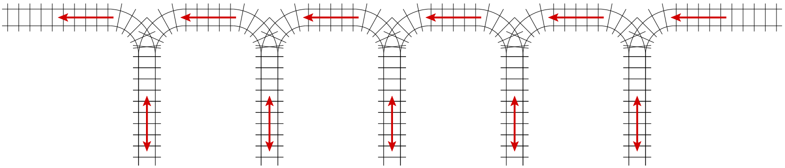

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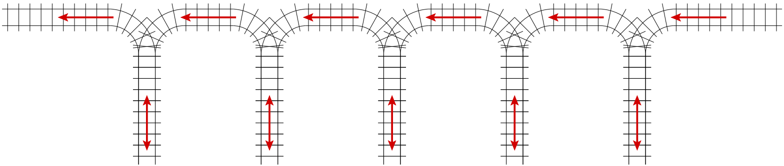

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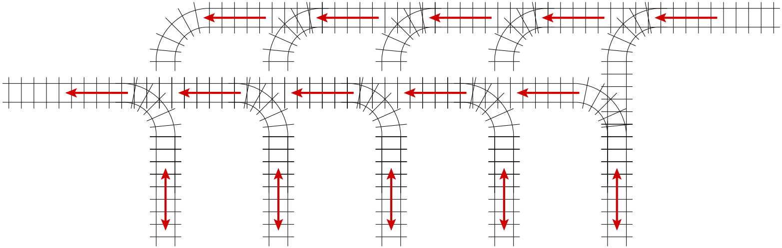
- mixed in and out: circle graphs



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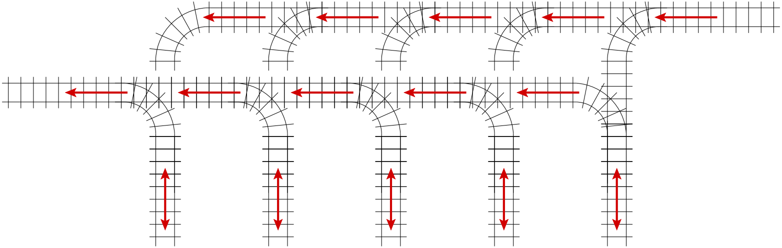

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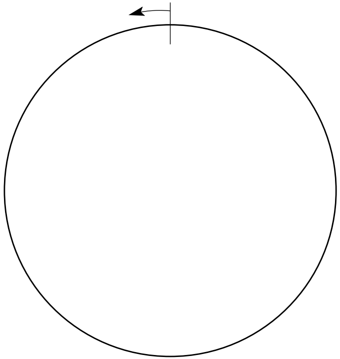
Relation to Circle Graphs

- $\pi = 24361875$
- π along circle



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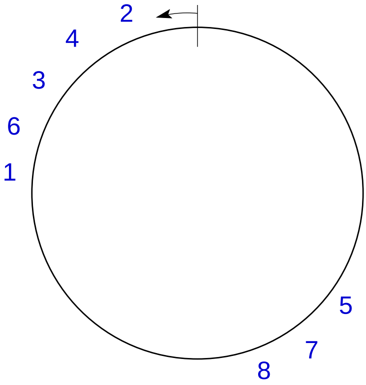

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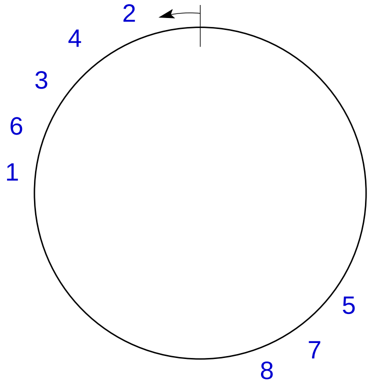

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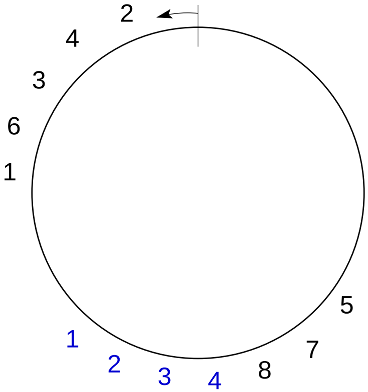

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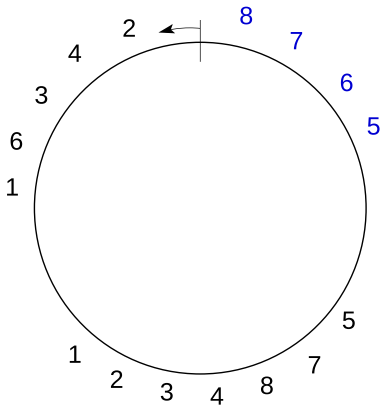

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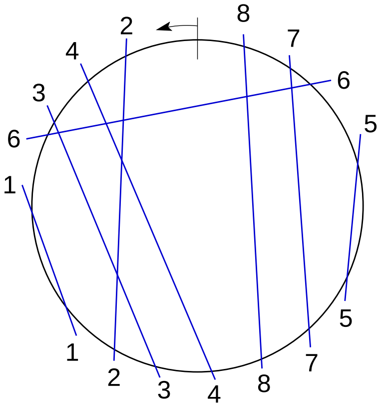

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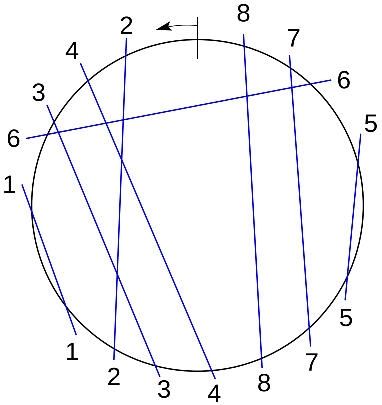
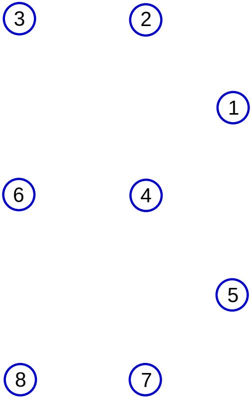


- associate chords with nodes
- nodes adjacent iff chords cross



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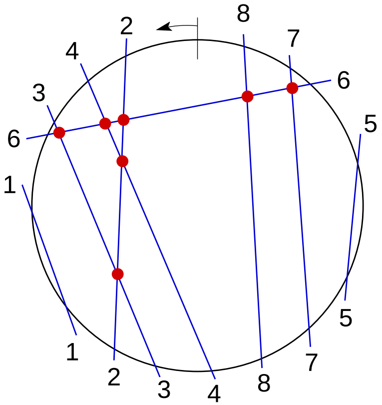

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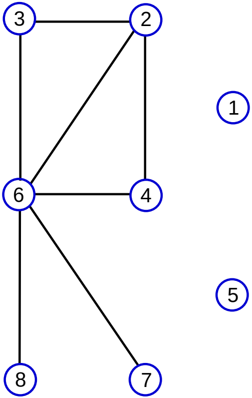

Imperfect Colorings

MIN-k-PARTITION

- graph G
- find a k-coloring $c: V(G) \to \{1, \ldots, k\}$
- goal: minimize number of monochromatic edges

- NP-hard to $\mathcal{O}(n^{2-\varepsilon})$-approximate for $k \geq 4$ Kann et al. (1997)

note that

- minimizing monochromatic edges does not minimize shuffles
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Theorem 1

For $k \geq 4$ and any $\varepsilon > 0$, it is *NP*-hard to approximate the minimum number of shuffles within $\mathcal{O}(n^{1-\varepsilon})$.

Theorem 2

Given a ρ-approximation algorithm for coloring circle graphs, there is a $3n$-approximation algorithm using $\rho k + 1$ stacks for $k \geq 3$.

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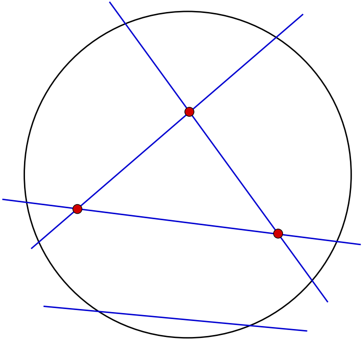
Proof of Theorem 1

Lemma 1

For $k \geq 4$ and any $\varepsilon > 0$, it is NP-hard to approximate MIN-k-PARTITION for circle graphs within $\mathcal{O}(n^{2-\varepsilon})$.

Proof :

- $I = (G, k)$ instance of k-COLORABILITY
- construct instance $J = (H, k)$ of MIN-k-PARTITION



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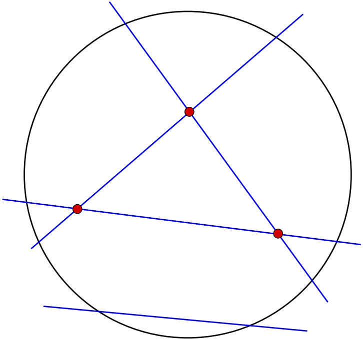
Proof :

- $s := n^{\frac{2}{\varepsilon}-1}$ copies $v_1, \ldots, v_s \;\; \forall v \in V(G)$
- $(v, w) \in E(G)$ $\Rightarrow (v_i, w_j) \in E(H) \;\; \forall i, j$



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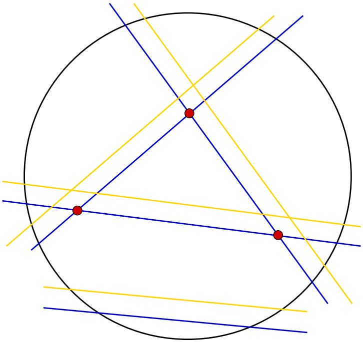

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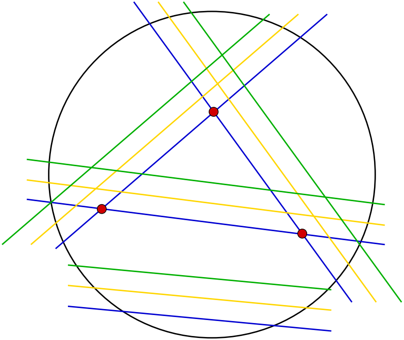

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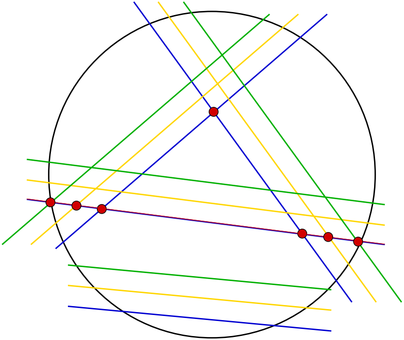

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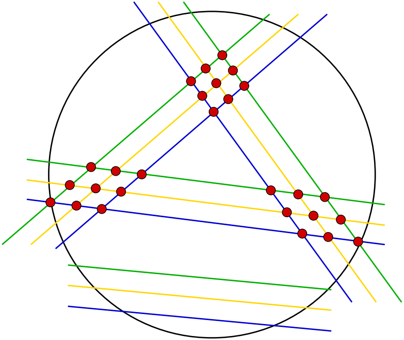

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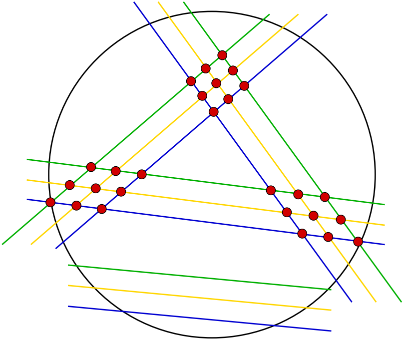

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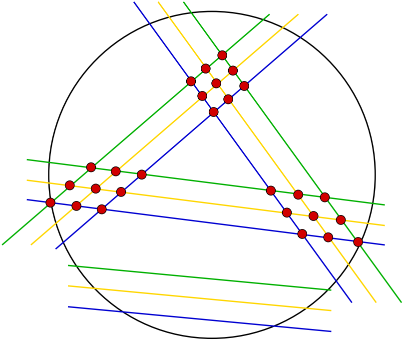

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Proof (cont'd):

- G k-colorable $\Leftrightarrow \gamma^*(H) = 0 \Leftrightarrow \gamma^*(H) < 1$
- G not k-colorable $\Leftrightarrow \gamma^*(H) \geq s^2 = n^{\frac{4}{\varepsilon}-2} = n^{\frac{2}{\varepsilon}(2-\varepsilon)} = N^{2-\varepsilon}$
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For a solution to an instance of STACK SORTING with L shuffles and the corresponding coloring c, the number γ of monochromatic edges in c is bounded by

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Proof:

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- a shuffle "resolves" at most $n-1$ monochromatic edges

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$\rightsquigarrow$ solution with L shuffles, corresponding coloring with γ monochromatic edges

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Given a ρ-approximation algorithm for coloring circle graphs, there is a $3n$-approximation algorithm using $\rho k + 1$ stacks for $k \geq 3$.

- iteratively find longest ρk-colorable prefix of items
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Stack Sorting with Conflicts

- conflict graph D
- $V(D) = \{1,\ldots,n\}$, $(i,j) \in E(D) \iff i$ not on top of j

Theorem 3

For any fixed $k \geq 3$, deciding feasibility of STACK SORTING WITH CONFLICTS is PSPACE-complete.

- uses nondeterministic constraint logic Hearn & Demaine (2005)
- reduction from CONFIGURATION TO EDGE

Open Problems

- first algorithmic study of sorting with stacks

open problems:

- close logarithmic gap between lower and upper bound
- complexity for $k = 2, 3$ stacks
- …

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