Single-vehicle Preemptive Pickup and Delivery Problem

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Agenda

Agenda

1. Definition of the problem
2. Representations of the solution - Complexity results
3. Formulation of the unitary case
4. Formulation of the SPPDP

Single-vehicle Preemptive Pickup and delivery Problem (SPPDP)

Input

- Digraph $D = (V, A)$
- depot $v_0 \in V$
- Cost vector $c \in \mathbb{R}^A$ associated with arcs
- k pairs (o^p, d^p), $p = 1, \ldots, k$
- k demands of transportation $q^1, \ldots, q^k$
- Vehicle with limited capacity B

Single-vehicle Preemptive Pickup and delivery Problem (SPPDP)

Objective

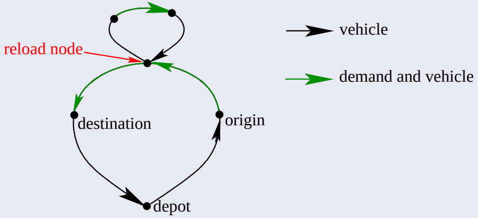
minimizing the vehicle trip cost so that

- The vehicle begins and ends at the depot
- Each arc is used at most once
- Demands are carried from their origin to their destination
- Capacity of the vehicle must not be exceeded
- Transportation with preemption

Variant of the SPDP

Transportation using preemption

Demands can be temporary unloaded anywhere.



Preemptive version of the problem.

Remark

No cost nor constraints associated with reloads.

Agenda

Solutions

Differences with the non-preemptive version

- The vehicle closed walk cannot be only defined by its arc set.
- Demand paths cannot be deduced from the vehicle closed walk.

A solution is characterized by

- Closed walk of the vehicle
  - Set of arcs
  - Sequence of arcs
- Demand paths
  - Set of arcs

Information necessary to define a solution

Reducing the number of variables

Can we discard some information ?

Possible only if we can compute the discarded information to obtain a feasible solution or attest there does not exist such discarded information.

Can we discard the following information ?

- arc sets associated with the demand paths
- Sequence of arcs of the vehicle closed walk

Can we discard the arc sets of the demands paths ?

Demand paths checking problem (simplified version)

- Input
  - Eulerian closed walk C on an Eulerian digraph $D = (V, A)$,
  - k pairs (o^i, d^i), $i = 1, 2, \ldots, k$, on V,
- Do there exist k arc-disjoint paths $L_1, L_2, \ldots, L_k$ so that
  - L_i is a $o^i d^i$-path $(i = 1, 2, \ldots, k)$,
  - for each path, the arcs are traversed in the same order as in C ?

Can we discard the arc sets of the demands paths ?

Theorem

The demand paths checking problem is NP-complete

Proof

Reduction from the arc-disjoint paths problem in acyclic digraphs

Consequences for the SPPDP

Information relative to the arc set of the demand paths is necessary

Can we discard the sequence of arcs of the vehicle closed walk ?

The Eulerian closed walk with precedence path constraints problem (ECWPPCP)

- Input
  - Eulerian digraph $D = (V, A)$
  - $v_0 \in V$
  - k paths on D
- Does there exist an Eulerian closed walk on D satisfying the precedence constraints induced by the simple paths ?

Results

Theorem

- ECWPPCP is NP-complete in general,
- Polynomial-time solvable if K Yout-free ou Yin-free.

Proof of the NP-completeness of the ECWPPCP

Reduction from

Directed Hamiltonian Circuit of indegrees and outdegrees exactly two Problem (2DHCP) :
Let $D_H = (V_H, A_H)$, $V_H = \{v_1, v_2, \ldots, v_n\}$, be a digraph so that $|\delta^+(v)| = |\delta^-(v)| = 2$ for every v. Does there exist a Hamiltonian circuit in D_H ?

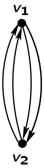
D_H contains n vertices

D contains :

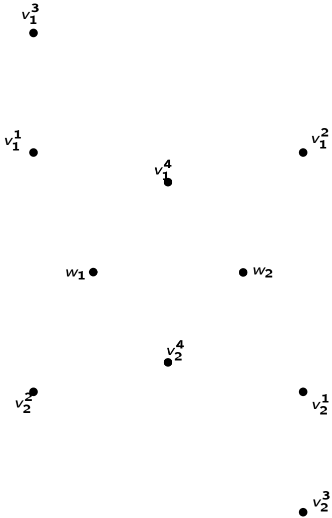
- $4n + 2$ vertices
- $10n + 2$ arcs

K contains $2n + 1$ paths

Example of construction

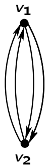


(a) digraph D_H : Input of 2DHCP



(b) digraph D : Input of ECWPPCP

Example of construction



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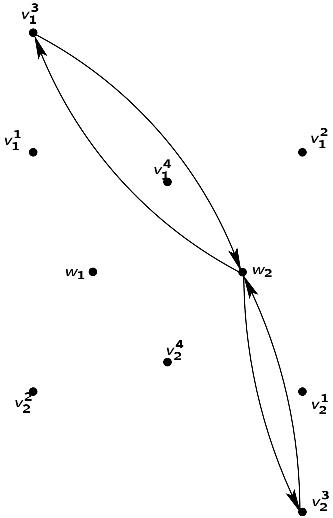
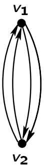


(b) digraph D : Input of ECWPPCP

Example of construction



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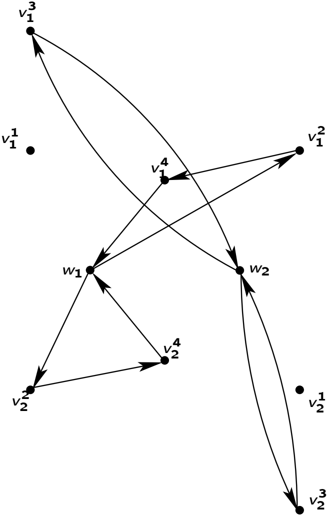
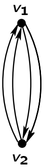


(b) digraph D : Input of ECWPPCP

Example of construction



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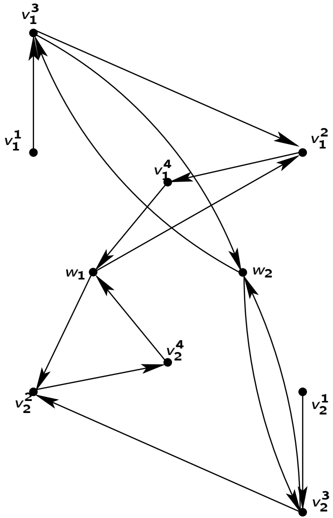
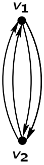


(b) digraph D : Input of ECWPPCP

Example of construction



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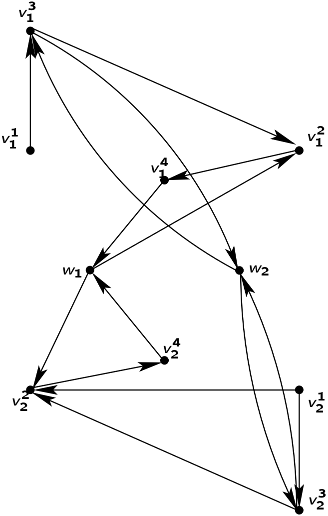
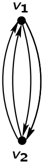


(b) digraph D : Input of ECWPPCP

Example of construction



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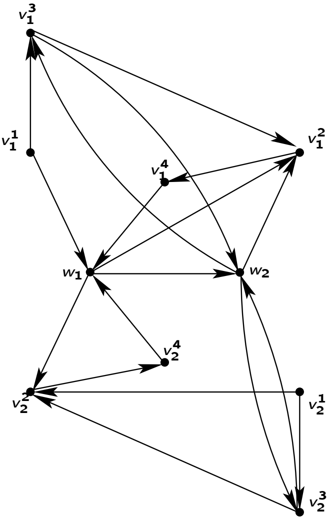
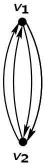


(b) digraph D : Input of ECWPPCP

Example of construction



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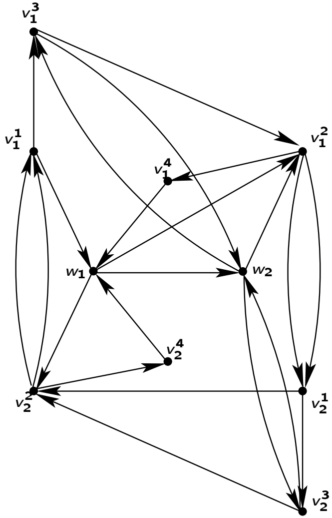
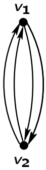


(b) digraph D : Input of ECWPPCP

Example of construction



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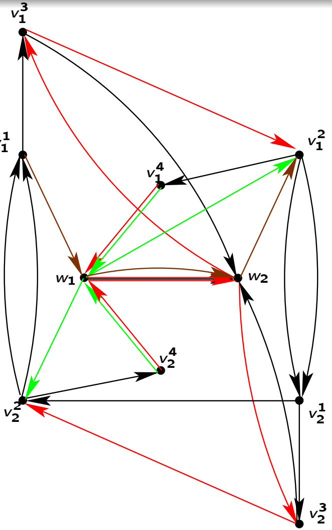


Starting vertex $v_0 = v_1^2$

(b) digraph D : Input of ECWPPCP

Polynomial-time solvable case

Hypothesis : the vehicle carries one demand at the same time

Definition

K Yout-free if every arc has at most one successor in K

Proposition

K Yout-free. Let $P = (a_1, a_2, \ldots, a_k)$, $k \geq 1$ be an open walk respecting K and v be the head of P. Then, there exist $a \in \delta^+(v)$ so that $(a_1, a_2, \ldots, a_k, a)$ respects K.

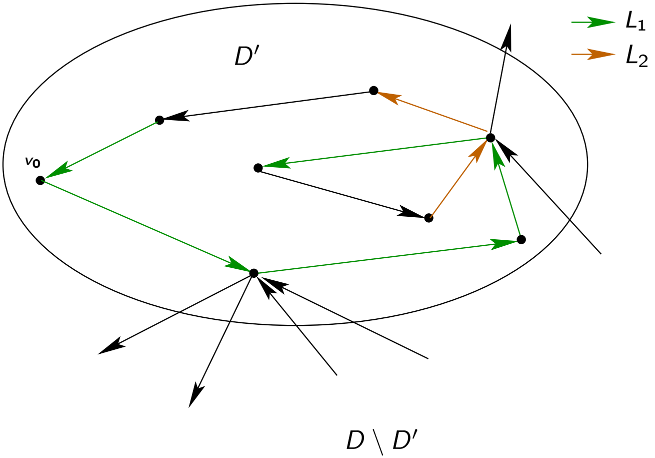
Polynomial-time solvable case

Definition of the Impregnable Eulerian Subdigraph (IES)

Let D' be an Eulerian subdigraph of D. $v \in V'$ is said D'-impregnable iff, for every $a \in \delta^{\text{out}}_{D'}(v)$, there exists $a' \in \delta^{\text{in}}_{D'}(v)$ so that

- $a' \prec_K a$ if $v = v_0$,
- either $a' \prec_K a$ or either v is incident with no arc of $A \setminus A'$, if $v \neq v_0$.

D' is said impregnable iff v is D'-impregnable for all $v \in V'$


L_1
L_2
D'
v_0
$D \setminus D'$

Algorithm of the ECWPPCP

Input : (D, v_0, K) with K Yout-free
Output : Feasible solution for the ECWPPCP or impregnable Eulerian subdigraph
1 - Current closed walk $C = \emptyset$ (C respects K)
2 - As long as possible
    Find closed walk C' (possible if non-D-impregnable vertex)
    Combine C' with C
    Remove of D arcs of C'
3 - If $A = \emptyset$ then feasible solution else IES

Theorem

If K Yout-free, then ECWPPCP has a feasible solution iff (D, v_0, K) does not contain any impregnable Eulerian subdigraph

Proof

($\Rightarrow$) Definition of impregnable Eulerian subdigraph
($\Leftarrow$) Consequence of Algorithm

Corollary

If K Yin-free, then ECWPPCP has a feasible solution iff (D, v_0, K) does not contain impregnable Eulerian subdigraph

Consequences for the SPPDP

Unitary case

Solution can be represented by

- Set of arcs of the vehicle closed walk
- Sets of arcs of the demand paths

General case

Solution can be represented by

- Set of arcs of the vehicle closed walk
- Sequence of arcs of the vehicle closed walk
- Sets of arcs of the demand paths

Agenda

Variables

unitary SPPDP : the vehicle can carry one demand at the same time

- Volume of the demands: $q^p = 1$ for all $p \in P$
- Capacity of the vehicle : $B = 1$

Variables

- $x_a^p = \begin{cases} 1 & \text{if the demand } p \text{ is carried on arc } a, \\ 0 & \text{otherwise,} \end{cases}$
  for all $a \in A$ and for all $p \in P$
- $y_a = \begin{cases} 1 & \text{if the vehicle traverses the arc } a, \\ 0 & \text{otherwise,} \end{cases}$
  for all $a \in A$

Valid constraints

The digraph induced by the vehicle closed walk is Eulerian

$$\sum_{a \in \delta^{\mathrm{out}}(W)} y_a - y_{a'} \geq 0 \qquad \forall\ W \subset V \text{ with } v_0 \in W,\ \forall\ a' \in A(\overline{W}) \tag{1}$$

$$\sum_{a \in \delta^{\mathrm{out}}(v)} y_a - \sum_{a \in \delta^{\mathrm{in}}(v)} y_a = 0 \qquad \forall\ v \in V \tag{2}$$

Valid constraints

Every demand is carried throug one path

$$\sum_{a \in \delta^{\mathrm{out}}(v)} x_a^p - \sum_{a \in \delta^{\mathrm{in}}(v)} x_a^p = b_v^p \quad \forall\, p \in P, \ \forall\, v \in V \tag{3}$$

$$\sum_{a \in \delta^{\mathrm{out}}(v)} x_a^p + x_{o^p d^p}^p \leq 1 \quad \forall\, p \in P, \ \forall\, v \in V \setminus \{o^p, d^p\} \tag{4}$$

Demand paths are arc-disjoint

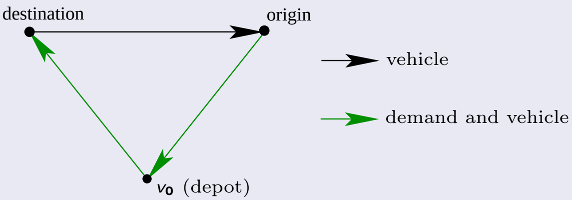
$$y_a - \sum_{p \in P} x_a^p \geq 0 \quad \forall\, a \in A \tag{5}$$

Precedence problem

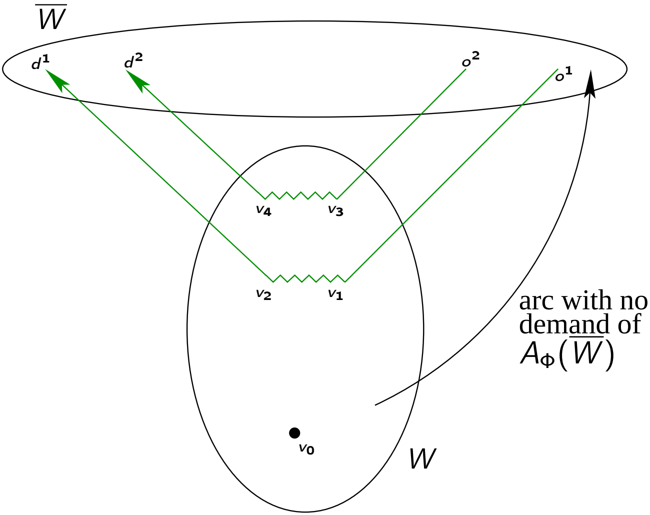
Remark

Constraints (1)-(5) are not sufficient

Example



Additional condition



Additional constraints

Vulnerability constraints

Let $W \subset V$ be so that $v_0 \in W$, $A_\Phi(\overline{W}) \neq \emptyset$, $\delta_\Phi(W) = \emptyset$. The vulnerability constraint associated with W

$$\sum_{a \in \delta^{\text{out}}(W)} y_a - \sum_{p \in A_\Phi(\overline{W})} \sum_{a \in \delta^{\text{out}}(W)} x_a^p \geq 1, \tag{6}$$

is valid for the unitary SPPDP.

Formulation of the unitary SPPDP

$$\mathcal{P}_1 = \{\min c^T y \mid (x, y) \in \{0, 1\}^n : (x, y) \text{ satisfy } (1) - (6)\}$$

Theorem

The unitary SPPDP is equivalent to $\mathcal{P}_1$

Constraints (1) are not necessary if arc costs are positive

Open question

Complexity of the separation problem of constraints (6)

Consequence : Complexity of the linear relaxation of $\mathcal{P}_1$ is an open question

Additional constraints

Relaxed vulnerability constraints

Let $W \subset V$ be so that $v_0 \in W$, $A_\Phi(\overline{W}) \neq \emptyset$, $\delta_\Phi(W) = \emptyset$. The relaxed vulnerability constraint associated with W

$$y(\delta^{\text{out}}(W)) - \sum_{p \in A_\Phi(\overline{W})} x^p(\delta^{\text{out}}(W)) + M \sum_{p \in A_\Phi(W)} x^p(\delta^{\text{out}}(W)) \geq 1, \quad (7)$$

is valid for the unitary SPPDP.

$$\mathcal{P}_2 = \{\min c^T y \mid (x, y) \in \{0,1\}^N : (x, y) \text{ satisfy } (1) - (5), (7)\}$$

Theorem

The unitary SPPDP is equivalent to $\mathcal{P}_2$

Separation problem of the relaxed vulnerability constraints

Theorem

Constraints (7) can be separated in polynomial time.

Algorithm

- Decomposition in $|P|$ subproblems
- Auxiliary digraph :
  - Contraction of the vertices o^p, d^p in v_p for all $p \in P$
  - Arc sets $A^p = \{(v_p, v) : v \in V(p)\}$ for all $p \in P$
  - $c_a = \begin{cases} +\infty & \text{if } a \in A^p \text{ for all } p \in P, \\ y_a - x_a(P) & \text{otherwise} \end{cases}$
- Computation of a $v_0 v_p$-minimum cut for all $p \in P$

Consequence

The linear relaxation of $\mathcal{P}_2$ is polynomial-time solvable.

Agenda

Solutions

SPPDP (general case)

Several demands can be carried at the same time
$q^p \in \mathbb{Z}_+$ for all $p \in P$ and $B \in \mathbb{Z}_+$ with $q^p \leq B$, for all $p \in P$

Information

- Arc sets of the demand paths
- Arc set of the vehicle closed walk
- Sequence (order) of arcs of the vehicle closed walk
  (Due to the NP-completeness of the ECWPPCP)

Solutions

Variables

- Same variables (x, y) as for the unitary case
- Order on the arcs of the vehicle closed walk may be represented with partial order (linear order on a subset of arcs)
  - Partial order may be represented using variables (y, η) with

    $$\eta_{aa'} = \begin{cases} 1 & \text{if } a \text{ is before } a' \text{ in the vehicle closed walk,} \\ 0 & \text{otherwise} \end{cases}$$

    for all pairs of distinct arcs $a, a' \in A$

Generalization of constraints (1)-(6)

Contraints (1)-(4)

unchanged

Capacity constraints

$$By_a - \sum_{p \in P} q^p x_a^p \geq 0 \tag{8}$$

for all arcs $a \in A$

Vulnerability constraints

$$\sum_{a \in \delta^{\mathrm{out}}(W)} y_a - \left\lceil \frac{1}{B} \sum_{p \in A_\Phi(\overline{W})} \sum_{a \in \delta^{\mathrm{out}}(W)} q^p x_a^p \right\rceil \geq 1 \tag{9}$$

for all vertex xubsets $W \subset V$ with $v_0 \in W$ and $A_\Phi(\overline{W}) \neq \emptyset$

Additional constraints

Partial order constraints

Ensure that (y, η) is a partial order

$$y_a + y_{a'} - \eta_{aa'} - \eta_{a'a} \leq 1 \qquad \forall\ a, a' \in A,\ \ a \neq a' \qquad (10)$$

$$\eta_{aa'} + \eta_{a'a} - y_a \leq 0 \qquad \forall\ a, a' \in A\ a \neq a' \qquad (11)$$

$$\eta_{aa'} + \eta_{a'a''} - \eta_{aa''} - y_{a'} \leq 0 \qquad \forall\ a \neq a' \neq a'' \in A \qquad (12)$$

Alternate constraints

Restrict partial orders to those corresponding to closed walks

$$\sum_{a \in \delta^{\text{out}}(v) \setminus \{a'\}} \eta_{aa'} - \sum_{a \in \delta^{\text{in}}(v)} \eta_{aa'} + y_{a'} = 0 \qquad \begin{array}{l} \forall v \in V \setminus \{v_0\}, \\ \forall a' \in \delta^{\text{out}}(v) \end{array} \qquad (13)$$

$$\sum_{a \in \delta^{\text{out}}(v_0) \setminus \{a'\}} \eta_{aa'} - \sum_{a \in \delta^{\text{in}}(v_0)} \eta_{aa'} = 0, \qquad \forall a' \in \delta^{\text{out}}(v_0) \qquad (14)$$

Additional constraints

Demand precedence constraints

In order to synchronize demand paths and vehicle closed walk

$$x_a^p + x_{a'}^p - \eta_{aa'} \leq 1 \qquad \begin{array}{l} \forall p \in P, \forall v \in V \setminus \{o^p, d^p\}, \\ \forall a \in \delta^{\mathrm{in}}(v), \forall a' \in \delta^{\mathrm{out}}(v) \end{array} \tag{15}$$

Formulation of the SPPDP

$$\mathcal{P} = \{\min c^T y \ : \ \{(x, y, \eta) \in \{0,1\}^n \text{ satisfait } (2)\text{-}(4), (8), (10)\text{-}(15)\}$$

Theorem

The SPPDP is equivalent to $\mathcal{P}$

Constraints (1) and (9) are redondant

Remark

The linear relaxation of $\mathcal{P}$ is polynomial-time solvable

Conclusion

Conclusion

- New complexity results
- New formulations with polynomial-time solvable linear relaxations

Perspectives

- Polyhedral study of the two formulations
  Theorem : Constraints (4)-(6) and trivial constraints define facets
- Branch-and-cut algorithms