How hard is it to find extreme Nash equilibria in network congestion games?

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13th Combinatorial Optimization Workshop
Aussois, January 2009

Talk Outline

1. Problem Formulation
2. Preliminary Results
3. Complexity Results for Worst Nash Equilibria
4. Complexity Results for Best Nash Equilibria

The model

- A directed graph $G(V, E)$ with multiple edges
- A source s and a sink t
- Non-decreasing latency functions $\ell_e : \mathbb{N}_0 \rightarrow \mathbb{R}_0^+$
- N users, each routing the same amount of unsplittable flow
- Strategy set for all users: $\mathcal{P}$ — set of all simple s-t-paths

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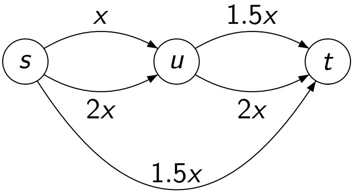
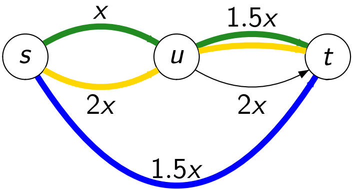


A flow is a function $f : \mathcal{P} \to \mathbb{N}_0$. The latency on a path $P \in \mathcal{P}$ is the sum of the latencies on its edges, i.e.,

$$\ell_P(f) := \sum_{e \in P} \ell_e \left(\sum_{P \in \mathcal{P}:\ e \in P} f_P \right)$$

Given a flow f the social cost are given by

$$C_{\max}(f) := \max_{P \in \mathcal{P}: f_P > 0} \ell_P(f).$$



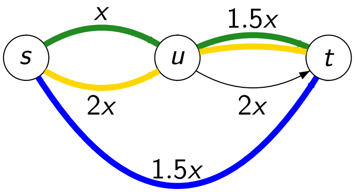
$$C_{\max}(f) = \max\{1+3, 2+3, 1.5\} = 5$$

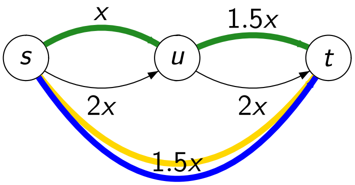
Nash Equilibrium

Definition (Nash Equilibrium)

A flow f is a Nash equilibrium, iff for all paths P_1, P_2 with $f_{P_1} > 0$ we have

$$\ell_{P_1}(f) \leq \ell_{P_2}(\tilde{f}) \text{ with } \tilde{f}_P = \begin{cases} f_P - 1 & \text{if } P = P_1 \\ f_P + 1 & \text{if } P = P_2 \\ f_P & \text{otherwise} \end{cases}.$$





Roughgarden Model

Network Congestion Game	Roughgarden
single-commoditiy unsplittable, unweighted makespan	multicommodity splittable sum

Existence of Nash equilibria

Existence of Nash equilibria

Theorem (Roughgarden and Tardos (2002))

The Nash flows of an instance are precisely the optima of a non-linear convex programming problem.
If f and $\tilde{f}$ are Nash flows then $\ell_e(f) = \ell_e(\tilde{f})$ for all $e \in E$. Hence, all Nash equilibria have the same social cost.

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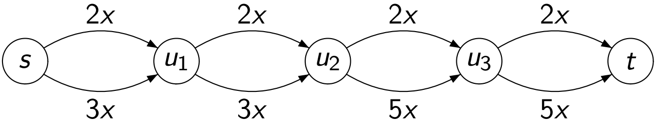
Theorem (Fabrikant et al. (2004))

Given a network congestion game the optimal solution of the following min-cost flow problem MCF(G) yields a Nash equilibrium:
For every edge $e \in E$ we need N copies with costs $c_{e_i} = \ell_e(i)$, $i = 1, \ldots, N$.
The capacities of all edges are 1 and we send N units of flow from s to t.

Consider the following instance with $N = 2$:



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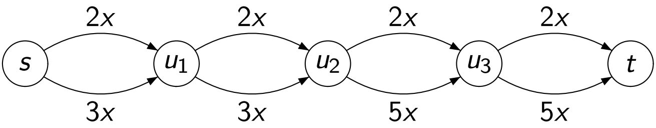
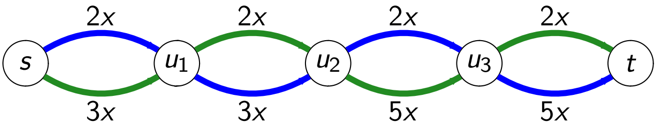


The solution with minimum social cost of 12 is given by



Extreme Nash equilibria

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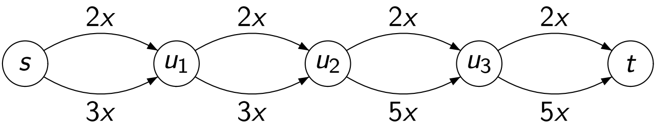

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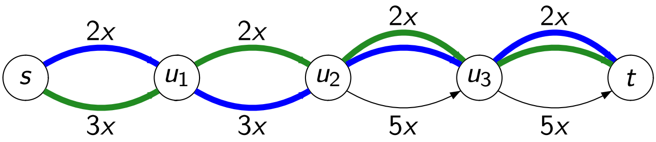

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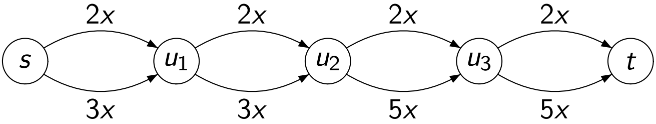

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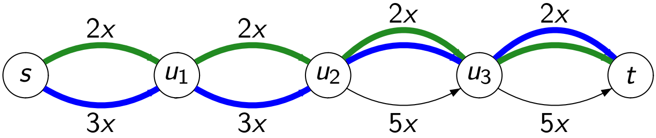


Worst Nash Equilibrium (W-NE for short):

Given: Network congestion game $(G = (V, E), (\ell_e)_{e \in E}, s \in V, t \in V, N)$ and a number $K > 0$

Question: Does there exist a Nash equilibrium f such that $C_{\max}(f) \geq K$?

Best Nash Equilibrium (B-NE for short):

Given: Network congestion game $(G = (V, E), (\ell_e)_{e \in E}, s \in V, t \in V, N)$ and a number $K > 0$

Question: Does there exist a Nash equilibrium f such that $C_{\max}(f) \leq K$?

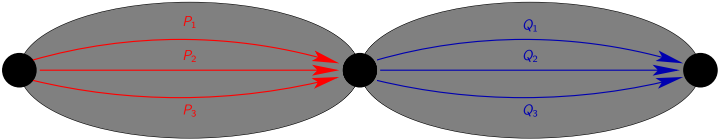
Unfortunately, it can be shown that in general neither a best nor a worst Nash equilibrium is an optimal solution of MCF(G).

Extreme Nash equilibria

Theorem (Fotakis(2002), Gairing(2005))

If the users have different weights and the graph G has only parallel links W-NE and B-NE are $\mathcal{NP}$-hard even for linear latency functions.

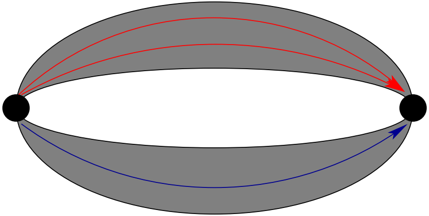
The series composition $G = S(G_1, G_2)$:



Lemma

Let f_i be a flow in G_i $(i = 1, 2)$. Let $f \in f_1 \otimes f_2$ then f is a Nash equilibrium in $S(G_1, G_2)$ if and only if f_i are Nash equilibria in G_i $(i = 1, 2)$.

The parallel composition $G = S(G_1, G_2)$:



Lemma

Let f_i be a flow in G_i $(i = 1, 2)$. Then $f = f_1 \cup f_2$ is a Nash equilibrium in $P(G_1, G_2)$ if and only if f_i are Nash equilibria in G_i $(i = 1, 2)$ and $C_{\max}(f_2) \leq L^+_{G_1}(f_1)$ and $C_{\max}(f_1) \leq L^+_{G_2}(f_2)$.

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For i = 1 to N do
     User i chooses a path with minimal latency
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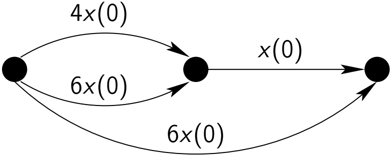

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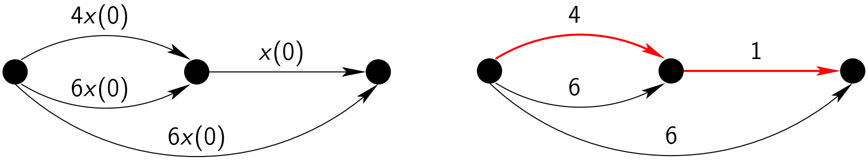

current makespan of user $1 = 5$

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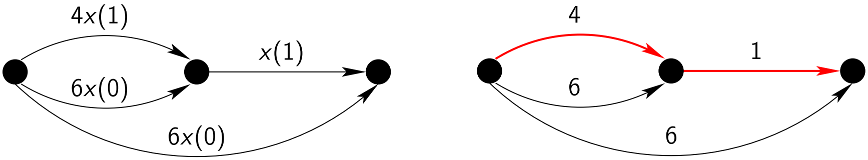

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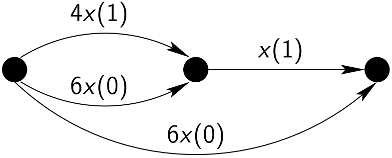

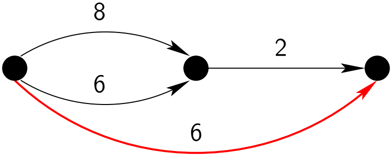

current makespan of user 1 = 5

current makespan of user 2 = 6

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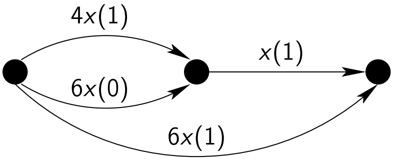

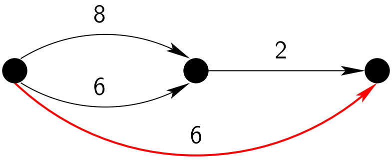

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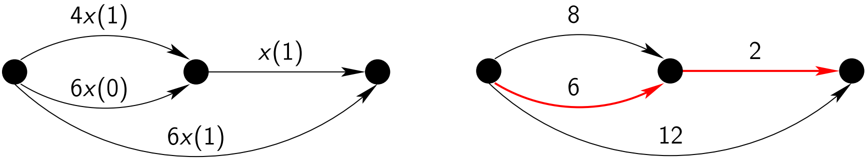

current makespan of user 1 = 6
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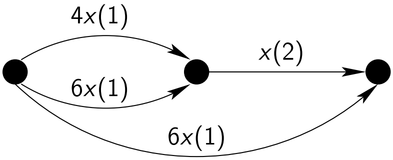

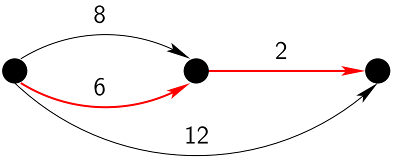

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The last user yields the maximum makespan!

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Worst Nash equilibria in SP-graphs

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Theorem (Fotakis (2006))

Greedy Best Response yields a Nash equilibrium in series-parallel graphs.

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Greedy Best Response yields a Nash equilibrium in series-parallel graphs.

Theorem (GHKSW(2008))

Greedy Best Response yields a worst Nash equilibrium in series-parallel graphs.

Worst Nash equilibria in arbitrary graphs

Theorem (GHKSW (2008))

Determining a worst Nash equilibrium is strongly NP*-hard even for two users on acyclic networks and with linear latency functions.*

Blocking Path Problem:

Given:	Digraph $G = (V, E)$ with source $s \in V$ and sink $t \in V$.
Question:	Does there exist an s-t-path $P \in \mathcal{P}$ such that after deleting the edges of P there is no path from s to t?

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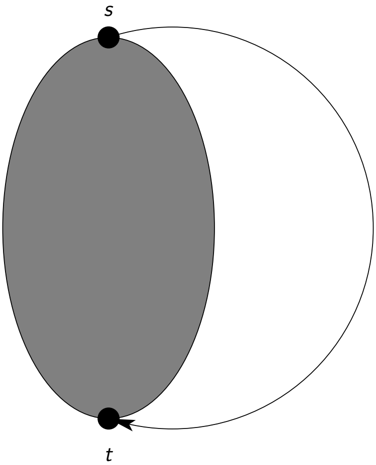
The Blocking Path Problem is strongly NP-hard even on acyclic networks.

Proof: Reduction from 3SAT.

Worst Nash equilibria in arbitrary graphs



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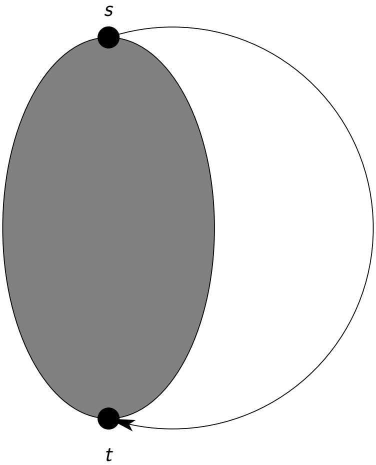
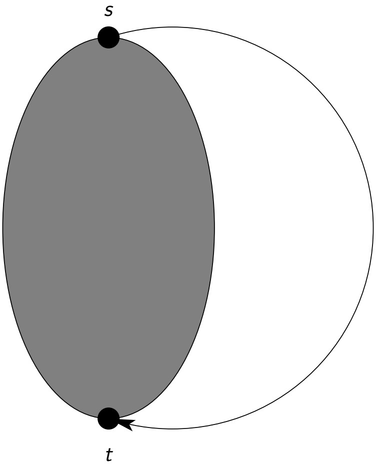


construct positive and integral edge lengths a_e such that every path from s to t has the same length L^*.



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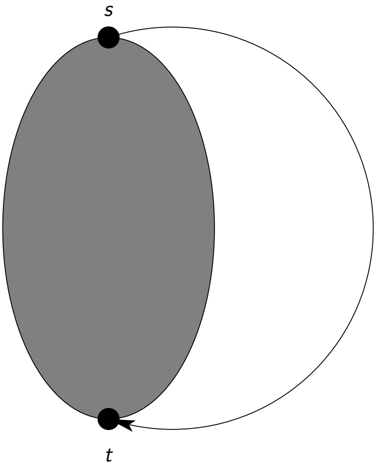
$$\ell_e(x) = \begin{cases} a_e x & \text{if } e \in E \\ (L^* + \frac{1}{2})(x) & \text{if } e = (s, t) \end{cases}$$



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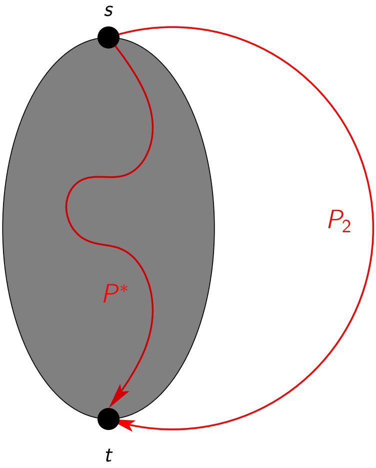
$$\exists \text{ blocking path } P^*$$

$$\Longleftrightarrow$$

$$\exists \text{ Nash equilibrium } f \text{ for two users with } C_{\max}(f) \geq L^* + \tfrac{1}{2}.$$



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$\exists$ Nash equilibrium f for two users with $C_{\max}(f) \geq L^* + \frac{1}{2}$.

Extreme Nash equilibria

	series-parallel graph	arbitrary graph
Worst NE	polynomially solvable (Greedy)	strongly NP-hard
Best NE		

Best Nash equilibrium

Best Nash Equilibrium (B-NE for short):

Given:	Network congestion game ($G = (V, E)$, $(\ell_e)_{e \in E}$, $s \in V$, $t \in V$, N) and a number $K > 0$
Question:	Does there exist a Nash equilibrium f such that $C_{\max}(f) \leq K$?

Best Nash equilibrium: N is part of input

Theorem (GHKSW (2008))

Determining a best Nash equilibrium is strongly NP*-hard even on series-parallel graphs and with linear latency functions if the number of users is part of the input.*

Numerical 3-Dimensional Matching:

Given: Disjoint sets X, Y, Z, each containing m elements, a weight $w(a)$ for all elements $a \in X \cup Y \cup Z$ and a bound $B \in \mathbb{Z}^+$.

Question: Does there exist a partition of $X \cup Y \cup Z$ into m disjoint sets $A_1, \ldots, A_m$ such that each A_j contains exactly one element from each of X, Y and Z and $\sum_{a \in A_i} w(a) = B$ for all i.

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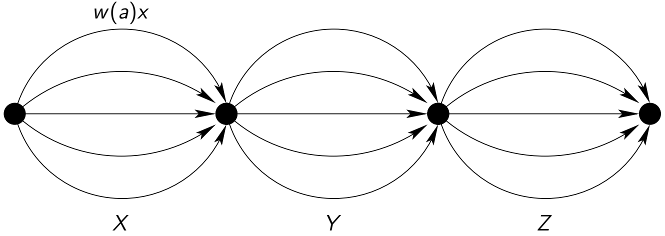
Assume w.l.o.g. that $w(a) \leq 2w(b)$ and $w(b) \leq 2w(a)$ for all $a, b \in X$ (Y, Z) holds.

Best Nash equilibrium: N is part of input



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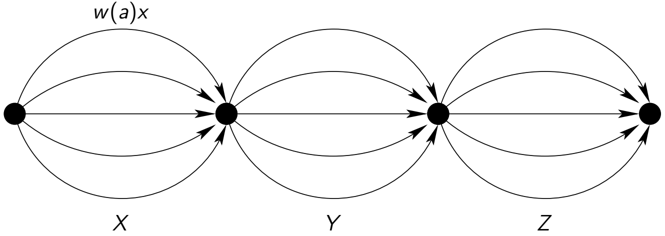

$$\exists \text{ numerical 3-dimensional matching}$$
$$\Longleftrightarrow$$
$$\exists \text{ Nash equilibrium } f \text{ for } m \text{ users with } C_{\max}(f) \leq B$$

Best Nash equilibrium: N is fixed

Theorem ([GHKSW (2008))

Determining a best Nash equilibrium is weakly NP*-hard even for two users on series-parallel graphs and with linear latency functions.*

Proof: Reduction from Even-Odd Partition Problem.

A dynamic programming algorithm

Let f be a Nash flow, then $C(f)$ denotes the set of latencies of the users with respect to f. $C(f)$ is called cost profile.

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$S_G(C)$ ... maximum latency for an additional user in a Nash flow in G with cost profile C.

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$S_G(C)$ ... maximum latency for an additional user in a Nash flow in G with cost profile C.

Idea: Find best C such that $S_G(C) < \infty$.

Best Nash Equilibrium: N is fixed

The series composition:

$$S_G(C) = \max_{C_1 \otimes C_2 \leq C} \{S_{G_1}(C_1) + S_{G_2}(C_2)\}$$

Best Nash Equilibrium: N is fixed

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The parallel composition:

$$S_G(C) = \max_{\substack{C_1 \cup C_2 = C \\ C_1 \leq S_{G_2}(C_2) \\ C_1 \leq S_{G_1}(C_1)}} \min\{S_{G_1}(C_1), S_{G_2}(C_2)\}$$

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☹ There is a huge number multisets C!
$\mathcal{O}((|V| \max_{e \in N} \ell_e(N))^N)$
$\implies$ pseudopolynomial-time algorithm for fixed N

Best Nash Equilibrium: N is fixed

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$\implies$ pseudopolynomial-time algorithm for fixed N

☺ Result is best possible!

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	series-parallel graph	arbitrary graph
Worst NE	polynomially solvable (Greedy)	strongly NP-hard
Best NE	strongly NP-hard if N is part of input weakly (!) NP-hard for fixed N	strongly NP-hard if N is part of input weakly (?) NP-hard for fixed N

Open Questions

- Can we give a bound on the price of anarchy for the network congestion games if the graph is series-parallel?
- What can be said about the price of stability for the network congestion games if the graph is series-parallel?

Thank you for your attention!