Combinatorial Abstractions for the Diameter of Polyhedra

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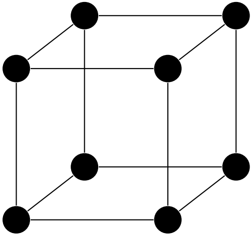
Outline

Diameter of a Polyhedron

- Let P be a pointed polyhedron.
- Vertices and edges of P define an undirected graph.
- $\mathrm{diam}(P)$ is the diameter of this graph.
- $\Delta(d, n)$ is the maximum diameter of d-dimensional polyhedra with n facets.



Diameter Problem

Problem

Find lower and upper bounds for $\Delta(d, n)$.

- Hirsch conjecture: $\Delta_b(d, n) \leq n - d$
  - true for $d \leq 3$ and $n - d \leq 5$ (Klee, Walkup 1967)
  - true for many special classes of polytopes
  - fails for unbounded polyhedra
- $n - d + \lfloor d/5 \rfloor \leq \Delta(d, n)$ (Klee, Walkup 1967)
- $\Delta(d, n) \leq n^{1+\log d}$ (Kalai, Kleitman 1992)
- $\Delta(d, n) \leq n2^{d-3}$ (Larman 1970)

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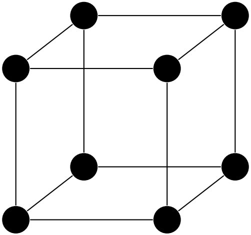
- Large gap between linear and superpolynomial.
- No progress in > 15 years – why?

Recipe for Abstractions

Lemma

$\Delta(d, n)$ is achieved by a simple polyhedron.



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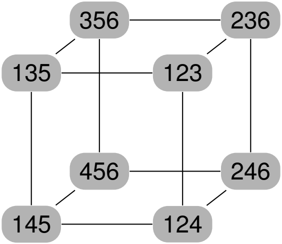

Base Abstraction

Our *base abstraction* is a graph $G = (V, E)$ with $V \subseteq \binom{[n]}{d}$ such that

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Dictionary

The elements of $[n]$ are called *symbols*, d is the *dimension*.

base abstraction	polyhedron
symbol	facet
set of symbols	face
set of d symbols	vertex
$D(d, n)$	$\Delta(d, n)$

Brief History Lesson

- Kalai (1992) mentions an abstraction with the additional property
$$(u, v) \in E \iff |u \cap v| = d - 1$$
- Adler, Dantzig (1974) studied an abstraction that in addition satisfies:
  - Every set of $d - 1$ symbols appears either in two vertices or in no vertex.

Our Results

- The best known general upper bound proofs translate to the base abstraction:
  - $D(d, n) \leq n^{1+\log d}$
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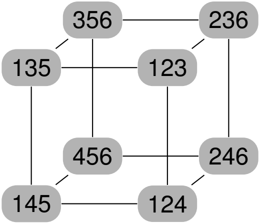
  - $D(d, n) \leq n^{1+\log d}$
  - $D(d, n) \leq n2^{d-1}$
- There is a superlinear lower bound:
  - $D(d, n) = \Omega(n^{3/2})$ when $d = \Theta(\sqrt{n})$.

Connected Layer Families

- Partition V into *layers* $\mathcal{L}_1, \ldots, \mathcal{L}_\ell$ such that
  - every set of symbols that is covered on layers i and j, $i < j$, is also covered on each layer in between.
- Such a partition is a *connected layer family*, ℓ is its *height*.
- Can partition an instance of the base abstraction using distance labels sucht that $\ell = \mathrm{diam}(G) + 1$:



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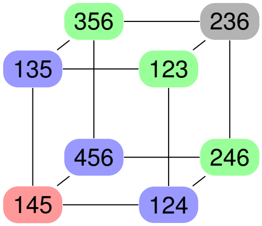

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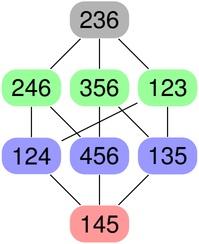

Equivalence

- We have seen: Every base abstraction yields a connected layer family.
- Now: Every connected layer family yields an instance of the base abstraction.

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145

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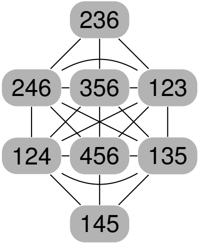

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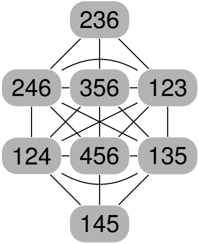


- We will construct a connected layer family of large height.

Outline

Diameter of Polyhedra and Abstractions

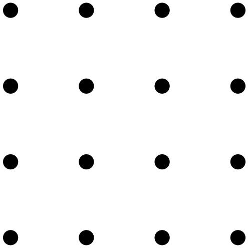
Disjoint Coverings

Lower Bound

Coverings

A simple example

Cover all points by as few 3-element sets as possible.



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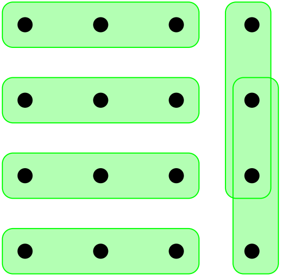
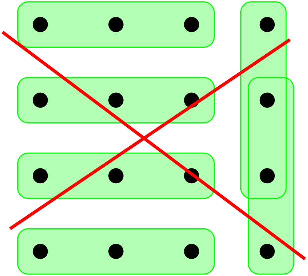


Coverings

A simple example

Cover all 2-element sets by as few 3-element sets as possible.



Coverings

- An (n, k, r)-covering of a set X of n elements is a collection of k-subsets of X that covers each r-subset of X at least once.
- $C(n, k, r)$ is the size of a smallest (n, k, r)-covering.

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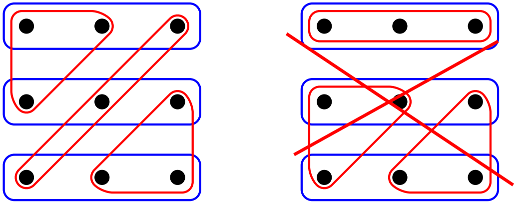
Theorem (Rödl 1985)

$\frac{C(n,k,r)}{\binom{n}{r}/\binom{k}{r}} \to 1$ for fixed k, r.

Families of Disjoint Coverings

- $DC(n, k, r)$ is the size of a largest family of pairwise disjoint (n, k, r)-coverings.

Example of Disjoint $(9, 3, 1)$-Coverings



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Every r-subset is contained in $\binom{n-r}{k-r}$ many k-subsets.
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Theorem

$DC(n, r+1, r) \geq n - r(r+2)$

Note: $DC(n, r+1, r) \leq n - r$

Disjoint $(n, r+1, r)$-Coverings

Theorem

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Proof.

- Use integers modulo n as set X.
- Create preliminary collections of $(r+1)$-element sets.

$$C_j = \{A \subset X \mid |A| = r+1, \sum_{a \in A} a = j\} \text{ for } j = 0 \dots n-1$$

- Every r-set is covered in exactly $n - r$ of the C_j.
- Use the Marriage Theorem to fill "holes" in $n - r(r+2)$ of the C_j, picking one $r(r+2)$-set from the other $r(r+2)$ collections for each "hole". □

Outline

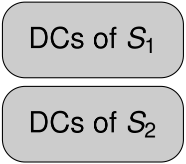
First Attempt: Disjoint Coverings

- Recall: $DC(n, r+1, r) \geq n - r(r+2)$
- Take a family of disjoint $(n, d, d-1)$-coverings $\mathcal{L}_1, \ldots, \mathcal{L}_{n-(d-1)(d+1)}$.
- This is a connected layer family of height $n - d^2 + 1$.
- No improved lower bound yet.

$\boxed{\text{DCs of } [n]}$

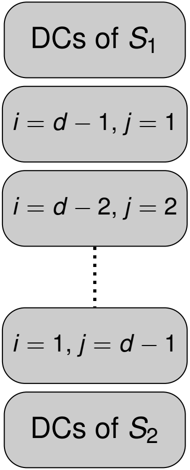
Second Attempt with Split Set of Symbol

- Instead of $[n]$, use two disjoint sets of symbols S_1 and S_2, $|S_1| = |S_2| = m$.
- Take separate families of disjoint $(m, d, d-1)$-coverings and concatenate them.
- Get a connected layer family of height $2(m - d^2 + 1)$.
- Height is slightly less than before, but now there are many unused potential vertices.



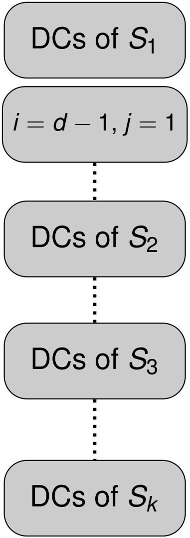
Mixing Sets of Symbols

- Add intermediate blocks for all $i, j > 0$ with $i + j = d$ as follows:
  - Disjoint $(m, i, i-1)$-coverings $A_0, \dots A_{k-1}$ of S_1
  - Disjoint $(m, j, j-1)$-coverings $B_0, \dots B_{k-1}$ of S_2
  - Form the q-th layer by combining sets from A_a with sets from B_b whenever $a + b = q \mod k$.
- Height is now $(d+1)(m - d^2 + 1)$.
- Almost $dn/2$, where n is the number of symbols.



Further Subdivision of the Set of Symbols

- Use k sets of symbols $S_1, \ldots, S_k$.
- Height is $\geq dkm - kd^3 - dm$.



Main Result and Open Problems

Theorem

Letting d grow as a function of n, $D(d, n) = \Omega(n^{3/2})$.

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Open Problems

- What geometric properties of polyhedra can be used to get better bounds on $\Delta(d, n)$?
- Is $D(d, n) \leq dn$?
- Find good lower bounds for $DC(n, k, r)$ when $k > r + 1$.

Fin

Thank you for your attention!