MEP123: Master Equality Polyhedron with one, two or three rows

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joint work with Sanjeeb Dash and Ricardo Fukasawa

Let $n, r \in \mathbb{Z}$ and $n \geq r > 0$.

MEP

$$K^1(n,r) = conv\left\{x \in \mathbb{Z}_+^{2n+1} : \sum_{i=-n}^{n} ix_i = r\right\}$$

- $K^1(n,r)$ was first defined by Uchoa, Fukasawa, Lysgaard, Pessoa, Poggi de Aragão and Andrade ('06) in a slightly different form.
- Using simple cuts based on $K^1(n,r)$, they reduce the integrality gap for capacitated MST instances by more than 50% on average.

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Gomory's MCGP

$$P^1(n,r) = conv\Big\{x \in \mathbb{Z}_+^{n} : -n x_{-n} + \sum_{i=1}^{n-1} i x_i = r\Big\}$$

Observation: MCGP is a lower dimensional face of MEP.

$$P^1(n,r) = conv\Big\{x \in \mathbb{Z}^n_+ : -nx_{-n} + \sum_{i \in I^G} ix_i = r\Big\}$$

where $I^G = [1, n-1] \equiv \{1, \ldots, n-1\}$.

Theorem (Gomory)

$\sum_{i \in I^G} \pi_i x_i \geq 1$ is a nontrivial facet defining inequality of $P^1(n,r)$ if and only if π is an extreme point of the following **polytope***:*

$$Q = \begin{cases} \pi_i + \pi_k \geq \pi_{(i+k)\, mod\, n} & \forall i,k \in I^G, \\ \pi_i + \pi_k = \pi_r & \forall i,k \in I^G,\ r = (i+k)\, mod\, n, \\ \pi_k \geq 0 & \forall k \in I^G, \\ \pi_r = 1. \end{cases}$$

A "Polar" description of MEP

Theorem (DFG)

$\sum_{i \in I} \pi_i x_i \geq 1$ is a nontrivial facet of $K^1(n,r)$ if and only if π is an extreme point of the following **polyhedron**:

$$T = \begin{cases} \pi_i + \pi_j & \geq & \pi_{i+j}, & \forall i,j \in I, & i+j \in I^+ \\ \pi_i + \pi_j + \pi_k & \geq & \pi_{i+j+k}, & \forall i \in I, & j,k,\ i+j+k \in I^+ \\ \pi_i + \pi_j & = & \pi_r, & \forall i,j \in I, & i+j = r \\ \pi_r & = & 1, & & \\ \pi_0 & = & 0, & & \\ \pi_{-n} & = & 0, & & \end{cases}$$

where $I = [-n, n]$ and $I^+ = [0, n]$.

- T and Q are not polars as they exclude trivial inequalities $x \geq 0$. (they also impose "complementarity" conditions $\pi_i + \pi_j = \pi_r$ for all $i + j = r$)
- Their extreme points give all nontrivial facets.
  - Q gives the convex hull of nontrivial facet coefficients (for MCGP)
  - T gives the convex hull *plus* some directions (for MEP).
- They can be used for efficient separation via linear programming.
- Not all facets of MEP can be obtained by lifting facets of MCGP.

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Regular subadditivity

$$\pi_i + \pi_j \geq \pi_{i+j} \qquad \forall i, j, i+j \in I = [-n, n]$$

Relaxed subadditivity

$$\begin{aligned} \pi_i + \pi_j &\geq \pi_{i+j}, & \forall i, j \in I, \quad & i+j \in I^+ = [0, n] \\ \pi_i + \pi_j + \pi_k &\geq \pi_{i+j+k}, & \forall i, j, k \in I, \quad & i+j+k \in I^+ \end{aligned}$$

- Regular subadditivity $\Rightarrow$ relaxed subadditivity
- All nontrivial facets satisfy regular subadditivity.
- If π satisfies either condition, then $\pi x \geq \pi_r$ is valid for $K^1(n, r)$.
- Subadditivity constraints introduce additional extreme points.

Regular subadditivity $\Rightarrow$ relaxed subadditivity:

$$T_{subadditivity} \subseteq T$$

$$T = \begin{cases} \text{relaxed } \textit{pairwise subadditivity} \\ \textit{complementarity} + \textit{normalization} \end{cases}$$

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$$T = conv.hull\{\underbrace{\pi_1, \ldots, \pi_k}_{\text{all non-trivial facets}}\} \quad + \quad \text{some unit directions}$$

$$T_{subadditivity} = conv.hull\{\underbrace{\pi_1, \ldots, \pi_k, \ldots, \pi_t}_{\text{all non-trivial facets and more}}\} + \text{ a smaller cone}$$

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Let $n \in \mathbb{Z}_+$, $r \in \mathbb{Z}_+^m$, $r \neq 0$ and $r \leq n\mathbf{1}$

MEP

$$K^m(n,r) = conv\Big\{x \in \mathbb{Z}_+^{|I|} : \sum_{i \in I} i x_i = r\Big\}$$

where $I = [-n, n]^m$.

MCGP

$$P^m(n,r) = conv\Big\{x \in \mathbb{Z}_+^{|I^+|} : \sum_{i \in I^+} i x_i = r \ (mod\ \mathbf{n})\Big\}$$

where $I^G = [0, n-1]^m \setminus \{\mathbf{0}\}$.

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Theorem (Gomory)

$\pi x \geq 1$ is a nontrivial facet defining inequality of $P^m(n,r)$ if and only if π is an extreme point of the following polytope:

$$Q^m = \begin{cases} \pi_i + \pi_k & \geq & \pi_{(i+k)\, mod\, n} & \forall i,k \in I^G, \\ \pi_i + \pi_k & = & \pi_r & \forall i,k \in I^G,\ r = (i+k)\, mod\, \mathbf{n}, \\ \pi_k & \geq & 0 & \forall k \in I^G \\ \pi_r & = & 1. & \end{cases}$$

$$K^m(n,r) = conv\Big\{x \in \mathbb{Z}_+^{|I|} : \sum_{i \in I} i x_i = r\Big\}$$

where $I = [-n,n]^m$ and let $I^+ = [0,n]^m \setminus \{\mathbf{0}\}$

Normalization

As the dimension of $K^m(n,r)$ is $|I| - m$, any inequality $\pi x \geq \beta$ can be normalized so that $\pi_i = 0$ for all $i \in I_N$, where

$$I_N = \left\{ \begin{bmatrix} -n \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -n \\ \vdots \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ 0 \\ \vdots \\ -n \end{bmatrix} \right\}$$

Theorem

After normalization all non-trivial facets can be written as $\pi x \geq 1$.

$$K^m(n,r) = conv\Big\{x \in \mathbb{Z}_+^{|I|} : \sum_{i \in I} i x_i = r\Big\}$$

where $I = [-n, n]^m$ and let $I^+ = [0, n]^m \setminus \{\mathbf{0}\}$

THEOREM

Generalizing the "non-trivial polar" T^1 for $K^1(n,r)$

$$T^m = \begin{cases} \sum_{i \in S} \pi_i & \geq & \pi_S, & \forall S \in \mathcal{S} \\ \pi_i + \pi_j & = & \pi_r, & \forall i,j \in I,\ i+j=r \\ \pi_0 = 0,\quad \pi_r & = & 1, & \pi_i = 0,\quad \forall i \in I_N \end{cases}$$

requires large S (some $|S| = O(n)$) if all $S \in \mathcal{S}$ satisfy $\sum_{i \in S} i \in I^+$.

- For MCGP, all $|S| = 2$; for MEP, all $|S| \leq 3$.

Definition

A *polaroid* T of $K^m(n, r)$ is a polyhedral set such that:

1. All $\pi \in T$, satisfy the normalization conditions
2. If $\pi \in T$ then $\pi x \geq 1$ is valid for all $x \in K^m(n, r)$
3. If $\pi x \geq 1$ is facet-defining for $K^m(n, r)$, then $\pi \in T$.

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NONTRIVIAL POLAR

- Polaroid $\subseteq$ Nontrivial Polar where

$$\text{Nontrivial Polar} = \{\pi \in \mathbb{R}^{|I|} \,:\, \pi x \geq 1 \text{ for all } x \in K^m(n,r)\}$$

$$\text{Nontrivial Polar} = conv.hull\{\underbrace{\pi_1, \ldots, \pi_k}_{\text{all non-trivial facets}}\} \quad + \quad \underbrace{\text{a cone}}_{\text{unit directions}}$$

$$\text{Polaroid} = conv.hull\{\underbrace{\pi_1, \ldots, \pi_k, \ldots, \pi_t}_{\text{all non-trivial facets and more}}\} + \text{ a smaller cone}$$

Separation via nontrivial polars

Definition

A *polaroid* T of $K^m(n,r)$ is a polyhedral set such that:

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3. If $\pi x \geq 1$ is facet-defining for $K^m(n,r)$, then $\pi \in T$.

Let P denote the continuous relaxation of $K^m(n,r)$.

Theorem

Given a point $x^* \in P$, and a polaroid T of $K^m(n,r)$. Then

1. *$x^* \in K^m(n,r)$ can be checked by solving an LP over T, and,*
2. *if $x^* \notin K^m(n,r)$ then a violated facet-defining inequality can be obtained by solving a second LP over T.*

T^1 IS A POLAROID FOR $K^1(n,r)$

$$T^1 = \begin{cases} \pi_i + \pi_j & \geq & \pi_{i+j}, & \forall i,j,i+j \in I \\ \pi_i + \pi_j & = & \pi_r, & \forall i,j \in I,\ i+j=r \\ \pi_0 = 0,\ \pi_r & = & 1, & \pi_{-n} = 0 \end{cases}$$

where $I = [-n,n]$

T^2 IS A POLAROID FOR $K^2(n,r)$

$$T^2 = \begin{cases} \pi_i + \pi_j & \geq & \pi_{i+j}, & \forall i,j,i+j \in I \\ \pi_i + \pi_j & = & \pi_r, & \forall i,j \in I,\ i+j=r \\ \pi_0 = 0,\ \pi_r & = & 1, & \pi_{\left[\begin{smallmatrix} -n \\ 0 \end{smallmatrix}\right]} = \pi_{\left[\begin{smallmatrix} 0 \\ -n \end{smallmatrix}\right]} = 0 \end{cases}$$

where $I = [-n,n]^2$

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T_a^3 IS **NOT** A POLAROID FOR $K^3(n,r)$

$$T_a^3 = \begin{cases} \pi_i + \pi_j & \geq & \pi_{i+j}, & \forall i,j,i+j \in I \\ \pi_i + \pi_j & = & \pi_r, & \forall i,j \in I,\ i+j=r \\ \pi_0 = 0,\ \pi_r & = & 1, & \pi_{\begin{bmatrix} -n \\ 0 \\ 0 \end{bmatrix}} = \pi_{\begin{bmatrix} 0 \\ -n \\ 0 \end{bmatrix}} = \pi_{\begin{bmatrix} 0 \\ 0 \\ -n \end{bmatrix}} = 0 \end{cases}$$

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T_b^3 IS **NOT** A POLAROID FOR $K^3(n,r)$

$$T_b^3 = \begin{cases} \pi_i + \pi_j + \pi_k & \geq & \pi_{i+j+k}, & \forall i,j,k,i+j+k \in I \\ \pi_i + \pi_j & = & \pi_r, & \forall i,j \in I,\ i+j=r \\ \pi_0 = 0,\ \pi_r & = & 1, & \pi_{\begin{bmatrix} -n \\ 0 \\ 0 \end{bmatrix}} = \pi_{\begin{bmatrix} 0 \\ -n \\ 0 \end{bmatrix}} = \pi_{\begin{bmatrix} 0 \\ 0 \\ -n \end{bmatrix}} = 0 \end{cases}$$

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$K^3(10, \mathbf{2})$

- Let
$$a = \begin{bmatrix} 10 \\ -10 \\ 10 \end{bmatrix} \quad b = \begin{bmatrix} -10 \\ 10 \\ 10 \end{bmatrix} \quad c = \begin{bmatrix} 1 \\ 1 \\ -9 \end{bmatrix}$$
and consider the point $\bar{x}$ and the inequality $\bar{\pi}x \geq 1$ where
  - $\bar{x}_a = \bar{x}_b = 1$, $\bar{x}_c = 2$ and all other $\bar{x}_i = 0$
  - $\pi_a = \pi_b = \pi_c = 0$ and all other $\pi_i = 1$ (including π_r)
- Note that, $\sum_{i \in I} i \cdot x_i = a + b + 2c = \mathbf{2}$ and $\bar{x} \in K^3(10, \mathbf{2})$.
- Also, $\bar{\pi}$ satisfies all 2 and 3-term subadditivity conditions:
  - $\pi_i + \pi_j \geq \pi_{i+j}$ for all $i, j, i + j \in I$,
  - $\pi_i + \pi_j + \pi_k \geq \pi_{i+j+k}$ for all $i, j, k, i + j + k \in I$,
- And yet, $\bar{\pi}\bar{x} = 0 \not\geq 1$!

Consider $K^m(n,r)$ and let $I = [-n,n]^m$.

K-term subadditivity

We say that $\pi \in \mathbb{R}^{|I|}$ satisfies k-term subadditivity if

$$\sum_{i \in S} \pi_i \;\geq\; \pi_S$$

for all $S \subseteq I$ such that (i) $|S| \leq k$ and (ii) $\sum_{i \in S} i \in I$

Validity via subadditivity

It is possible to construct invalid cuts $\pi x \geq 1$ for $K^m(n,r)$ where π satisfies the normalization conditions and k-subadditivity unless

$$k \geq \max\{2,\; 3 \cdot 2^{m-3} + 1\}$$

(for $m \geq 1$, the lower bound is: $2, 2, 4, 7, 13, 25, \ldots$)

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Theorem

If π satisfies 4-term subadditivity, then $\pi x \geq 1$ is valid for $K^3(n, r)$.

T^3 is a polaroid for $K^3(n, r)$

$$T^3 = \begin{cases} \pi_i + \pi_j + \pi_k + \pi_l & \geq & \pi_{i+j+k+l}, & \forall i, j, k, l, i+j+k+l \in I \\ \pi_i + \pi_j & = & \pi_r, & \forall i, j \in I,\ i+j=r \\ \pi_0 = 0,\ \pi_r & = & 1, & \pi_{\begin{bmatrix}-n\\0\\0\end{bmatrix}} = \pi_{\begin{bmatrix}0\\-n\\0\end{bmatrix}} = \pi_{\begin{bmatrix}0\\0\\-n\end{bmatrix}} = 0 \end{cases}$$

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Thank you...