An Efficient Algorithm for Partial Order Production

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Sorting by Comparisons

**Input:** a set $T$ of size $n$, totally ordered by $\leq$

**Goal:** place the elements of $T$ in a vector $v$ in such a way that

$$v[1] \leq v[2] \leq \cdots \leq v[n]$$

after asking a min number of questions of the form “is $t \leq t'$?”
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Partial Order Production ("Partial Sorting")

**Input:** a set $T$ of size $n$, totally ordered by $\leq$

a partial order $\preceq$ on the set of positions $[n] := \{1, 2, \ldots, n\}$

**Goal:** place the elements of $T$ in a vector $v$ in such a way that

$$v[i] \leq v[j] \quad \text{whenever } i \preceq j$$

after asking a min number of questions of the form “is $t \leq t'$?”
Particular Cases (1/2)

Heap Construction

```
```

or

```
v[1]
  └── v[2]
      └── v[4]
  └── v[3]
      └── v[6]
```

or

```
v[1]
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Find the elements of rank $r_1$, $r_2$, $\ldots$, $r_k$
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Target poset $P := ([n], \preceq)$ is a weak order
Particular Cases (2/2)

Multiple Selection

Find the elements of rank $r_1, r_2, \ldots, r_k$

Target poset $P := ([n], \preccurlyeq)$ is a weak order

$\exists$ near-optimal algorithm (Kaligosi, Mehlhorn, Munro and Sanders, 05)
Worst Case Lower Bounds

Well known fact. For Sorting by Comparisons:

\[ \text{worst case \#comparisons} \geq \lg n! \]
Fact. (Schönage 76, Aigner 81) For Partial Order Production:

\[
\text{worst case \#comparisons } \geq \log n! - \log e(P) =: LB
\]

where \( e(P) := \# \text{ linear extensions of } P \)
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\[
\geq \frac{n!}{8}
\]
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\text{worst case } \#\text{comparisons} \geq \ell g n! - \ell g e(P) =: LB
\]

where \( e(P) := \# \text{ linear extensions of } P \)

\[
|\text{leaf set}| \leq e(P) \implies \#\text{comparisons} \geq \ell g \frac{n!}{e(P)} = LB
\]
Problem History

1976 Schönage defined POP problem
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1981  Aigner studied POP problem
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1985 two surveys: Bollobás & Hell, and Saks.
Saks conjectured that ∃ algorithm for POP problem
s.t. worst case #comparisons = \( O(LB) + O(n) \)
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   Saks conjectured that ∃ algorithm for POP problem
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1989 Yao solved Saks’ conjecture, stated open problems
Our Result

There exists a $O(n^3)$ algorithm for the POP problem s.t.

\[
\text{worst case } \#\text{comparisons} = LB + o(LB) + O(n)
\]

Improvements over Yao’s algorithm:

- overall complexity is polynomial
- smaller number of comparisons
A Simple Plan

1. Extend the target poset $P$ to a weak order $W$

2. Solve the problem for $W$ using Multiple Selection algorithm
The entropy of $G = (V, E)$ equals:

$$H(G) := \min_{x \in STAB(G)} -\frac{1}{n} \sum_{v \in V} \log x_v$$

where $STAB(G) :=$ stable set polytope of $G$
Key Tool: the Entropy of a Graph

The entropy of $G = (V, E)$ equals:

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- Introduced in information theory by J. Körner (73)
- Graph invariant with lots of applications (mostly in TCS)
  - bounds for perfect hashing
  - circuit lower bounds for monotone Boolean functions
  - sorting under partial information (Kahn and Kim 95)
  - ...
Lemma. (Kahn and Kim 95)

\[-n \, H(G) \leq \lg \text{Vol}(STAB(G)) \leq n \, \lg n - \lg n! - n \, H(G)\]

\[\equiv \lg \text{Vol}(\text{Box})\]

\[\equiv \lg \text{Vol}(\text{Simplex})\]
Lemma. (Kahn and Kim 95)

\[-n H(G) \leq \lg \text{Vol}(\text{STAB}(G)) \leq n \lg n - \lg n! - n H(G)\]

\[\underbrace{\lg \text{Vol}(\text{Box})}_{=\lg \text{Vol}(\text{Box})} \leq \underbrace{\text{Vol}(\text{Simplex})}_{=\lg \text{Vol}(\text{Simplex})}\]
Lemma. (Kahn and Kim 95)

\[-nH(G) \leq \lg \text{Vol}(STAB(G)) \leq n \lg n - \lg n! - nH(G)\]

\[= \lg \text{Vol}(Box) \leq \lg \text{Vol}(Simplex)\]
Comparability Graphs and Entropy

\[ G(P) := \text{comparability graph of target poset } P \]
\[ H(P) := H(G(P)) \]

**Lemma.** (Stanley 86) \[ \text{Vol}\left( STAB\left( G(P) \right) \right) = \frac{e(P)}{n!} \]

**Corollary.** \[ n H(P) - n \log e \leq LB \leq n H(P) \]
Weak Order Extensions → Colorings

Observation.

*Every weak order extension* \( W \) *of* \( P \) *gives a coloring of* \( G(P) \)

\[ \Downarrow \]

Want: “good” coloring of \( G(P) \)

\( W \) extends \( P \) \( \implies \) \( STAB(G(P)) \supseteq STAB(G(W)) \)

\( \implies \) \( H(P) \leq H(W) \)

Intuition.

\( H(W) \) *should be as small as possible*

\[ \Downarrow \]

*The class sizes should be distributed as unevenly as possible*
Greedy Colorings and Greedy Points
For $G = \text{perfect graphs}$

Iteratively remove a maximum stable set from $G$

$\leadsto$ sequence $S_1, S_2, \ldots, S_k$ of stable sets

- Gives greedy coloring ($k$ colors, $i$th color class $= S_i$)
- Also gives greedy point:

$$\tilde{x} := \sum_{i=1}^{k} \frac{|S_i|}{n} \cdot \chi^{S_i} \in STAB(G)$$
Theorem. Let $G$ be a perfect graph on $n$ vertices and denote by $\tilde{g}$ the entropy of an arbitrary greedy point $\tilde{x} \in \text{STAB}(G)$. Then

$$\tilde{g} \leq \frac{1}{1 - \delta} \left( H(G) + \lg \frac{1}{\delta} \right)$$

for all $\delta > 0$, and in particular

$$\tilde{g} \leq H(G) + \lg H(G) + O(1).$$

Proof idea. Dual fitting, using min-max relation

$$H(G) + H(\bar{G}) = \lg n$$

due to Csiszár, Körner, Lovász, Marton and Simonyi (90) □
Colorings $\not\rightarrow$ Weak Order Extensions
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Colorings $\leftrightarrow$ Weak Order Extensions

$P \rightarrow$ colorings of $G(P)$ $\leftarrow \Rightarrow$ need to "uncross" our greedy colorings
Colorings $\not\rightarrow$ Weak Order Extensions

Weak order extensions of $P \rightarrow$ colorings of $G(P)$

$\implies$ need to “uncross” our greedy colorings
Uncrossing a Greedy Coloring

\[ D = D(P) := \text{auxiliary network with source } s, \text{ sink } t \]
\[ D = (N(D), A(D)) \]
(H-potential) \( \min \frac{1}{n} \sum_{v \in V} \lg x_v \)

s.t. \( x_v = y_{v^+} - y_{v^-} \quad \forall v \in V \)
\( y_a \leq y_b \quad \forall (a, b) \in A(D) \)
\( y_s = 0 \)
\( y_t = 1 \)

Find potential \( \tilde{y} \) for greedy point \( \tilde{x} \) (by DP)

We get:

- collection of open intervals \( \{ (\tilde{y}_{v^-}, \tilde{y}_{v^+}) \}_{v \in V} \)
- interval order \( I \) extending \( P \), with \( H(I) \) close to \( H(P) \)
Main Steps of our Algorithm

1. $P^{\text{greedy+DP}} \rightarrow I$

2. $I^{\text{greedy}} \rightarrow W$

3. Use Multiple Selection algorithm of Kaligosi et al. on $W$

**Theorem.** The algorithm above solves the POP problem, in $O(n^3)$ time, after performing at most

$$LB + o(LB) + O(n)$$

comparisons
Further Result & Open Questions

**Tightness result:**
- Any algorithm reducing the POP problem to Multiple Selection can be forced to perform

\[ LB + \Omega(n \lg \lg n) \]

comparisons for some \( P \) with \( H(P) \approx \frac{1}{2} \lg n \)

**Open questions:**
- Is there an algorithm performing \( LB + O(n) \) comparisons?
- What about Partial Order Production under Partial Information?
Thank You!

P.S.: The paper is available on ArXiv