An Efficient Algorithm for Partial Order Production

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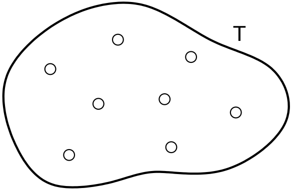
Sorting by Comparisons

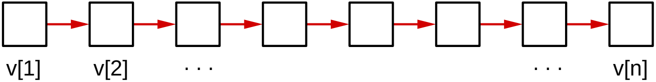
Input: a set T of size n, totally ordered by $\leqslant$

Goal: place the elements of T in a vector v in such a way that

$$v[1] \leqslant v[2] \leqslant \cdots \leqslant v[n]$$

after asking a min number of questions of the form "is $t \leqslant t'$?"





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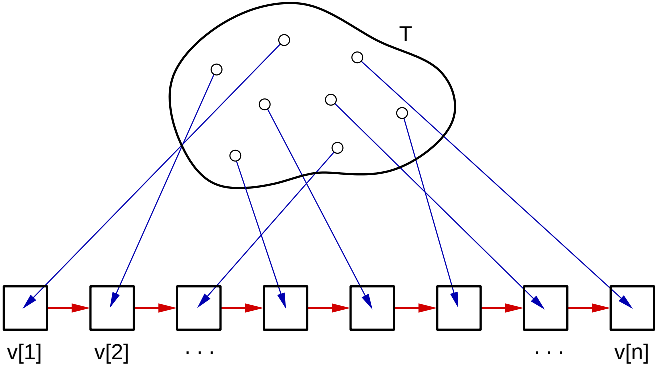


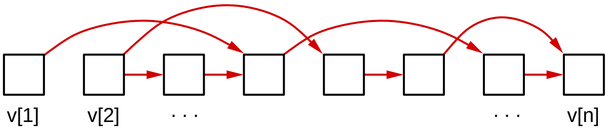
Partial Order Production ("Partial Sorting")

Input: a set T of size n, totally ordered by $\leqslant$
a partial order $\preccurlyeq$ on the set of positions $[n] := \{1, 2, \ldots, n\}$

Goal: place the elements of T in a vector v in such a way that

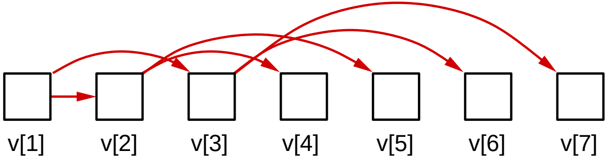
$$v[i] \leqslant v[j] \quad \text{whenever } i \preccurlyeq j$$

after asking a min number of questions of the form "is $t \leqslant t'$?"

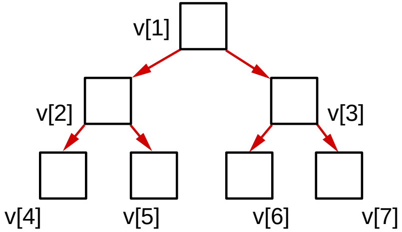


Particular Cases (1/2)

Heap Construction

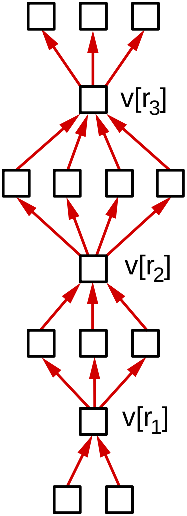


or



Particular Cases (2/2)

Multiple Selection



Find the elements of rank $r_1, r_2, \ldots, r_k$

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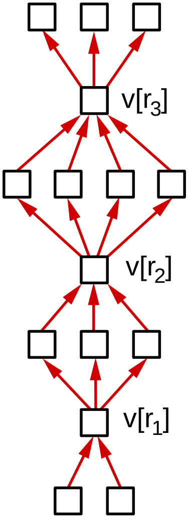

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Target poset $P := ([n], \preccurlyeq)$ is a weak order

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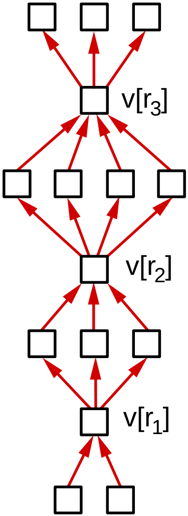

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Target poset $P := ([n], \preccurlyeq)$ is a weak order

$\exists$ near-optimal algorithm (Kaligosi, Mehlhorn, Munro and Sanders, 05)

Worst Case Lower Bounds

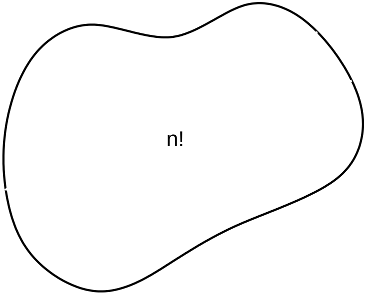
Well known fact. *For Sorting by Comparisons:*

$$\text{worst case \#comparisons} \geq \lg n!$$

Fact. (Schönage 76, Aigner 81) *For Partial Order Production:*

$$\text{worst case \#comparisons} \geq \underbrace{\lg n! - \lg e(P)}_{=:\, LB}$$

where $e(P) :=$ # linear extensions of P



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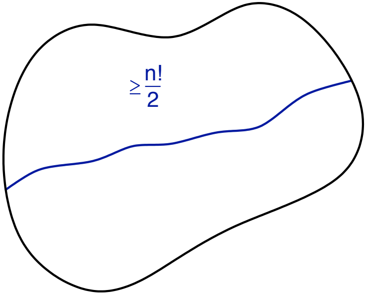

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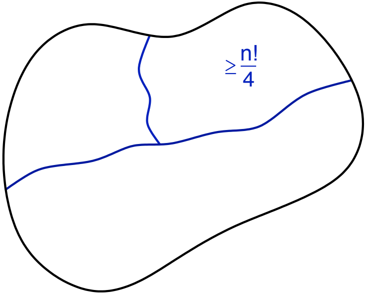

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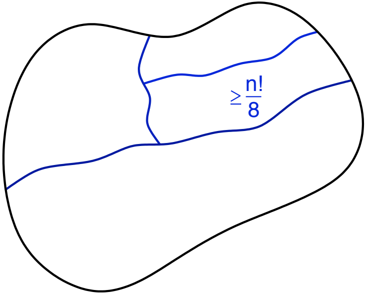

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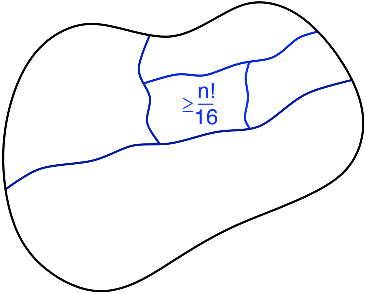

$$|\text{leaf set}| \leq e(P) \Longrightarrow \#\text{comparisons} \geq \lg \frac{n!}{e(P)} = LB$$

Problem History

1976 Schönage defined POP problem

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Saks conjectured that $\exists$ algorithm for POP problem
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1989 Yao solved Saks' conjecture, stated open problems

Our Result

There exists a $O(n^3)$ algorithm for the POP problem s.t.

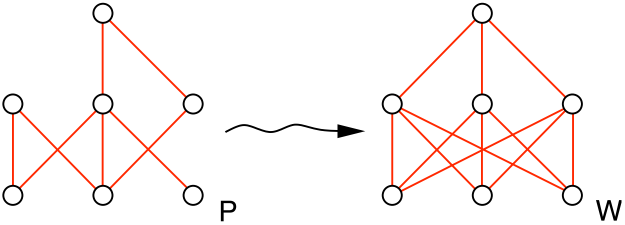
$$\text{worst case } \#\text{comparisons} = LB + o(LB) + O(n)$$

Improvements over Yao's algorithm:

- overall complexity is polynomial
- smaller number of comparisons

A Simple Plan

1. Extend the target poset P to a weak order W
2. Solve the problem for W using Multiple Selection algorithm



Key Tool: the Entropy of a Graph

The entropy of $G = (V, E)$ equals:

$$H(G) := \min_{x \in STAB(G)} -\frac{1}{n} \sum_{v \in V} \lg x_v$$

where $STAB(G) :=$ stable set polytope of G

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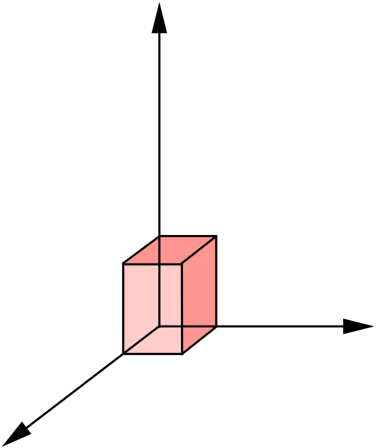
- Introduced in information theory by J. Körner (73)
- Graph invariant with lots of applications (mostly in TCS)
  - bounds for perfect hashing
  - circuit lower bounds for monotone Boolean functions
  - sorting under partial information (Kahn and Kim 95)
  - ...

Lemma. (Kahn and Kim 95)

$$\underbrace{-n\,H(G)}_{=\lg \mathrm{V}ol(Box)} \leq \lg \mathrm{Vol}(STAB(G)) \leq \underbrace{n \lg n - \lg n! - n\,H(G)}_{=\lg \mathrm{V}ol(Simplex)}$$



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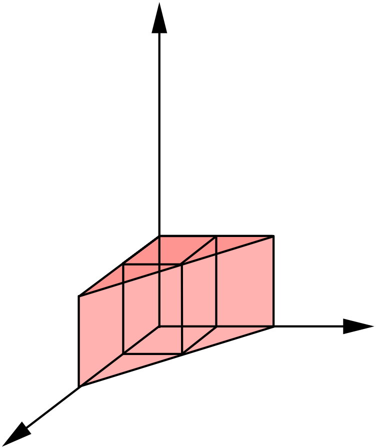

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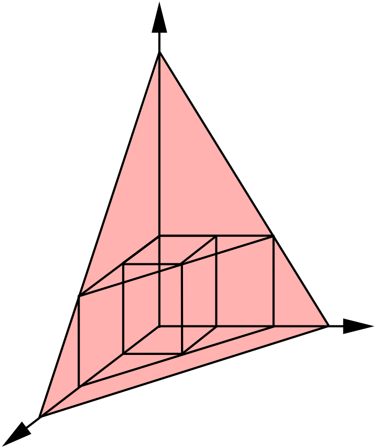
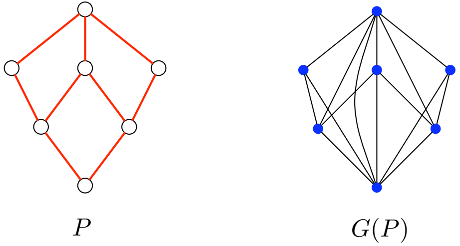


Comparability Graphs and Entropy

$G(P) :=$ comparability graph of target poset P

$H(P) := H(G(P))$



Lemma. (Stanley 86) $\mathrm{Vol}\Big(STAB\big(G(P)\big)\Big) = \dfrac{e(P)}{n!}$

Corollary. $n\,H(P) - n \lg e \leq LB \leq n\,H(P)$

Weak Order Extensions $\rightarrow$ Colorings

Observation.

Every weak order extension W of P gives a coloring of $G(P)$

$$\Downarrow$$

Want: "good" coloring of $G(P)$

$$\begin{aligned} W \text{ extends } P \quad &\Longrightarrow \quad STAB\big(G(P)\big) \supseteq STAB\big(G(W)\big) \\ &\Longrightarrow \quad H(P) \leq H(W) \end{aligned}$$

Intuition.

$H(W)$ should be as small as possible

$$\Downarrow$$

The class sizes should be distributed as unevenly as possible

Greedy Colorings and Greedy Points

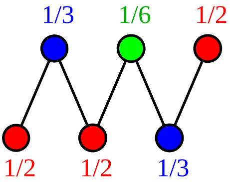
For G = perfect graphs

Iteratively remove a maximum stable set from G

$\rightsquigarrow$ sequence $S_1, S_2, \ldots, S_k$ of stable sets

- Gives greedy coloring (k colors, ith color class = S_i)
- Also gives greedy point:

$$\tilde{x} := \sum_{i=1}^{k} \frac{|S_i|}{n} \cdot \chi^{S_i} \quad \in \quad STAB(G)$$



Theorem. *Let G be a perfect graph on n vertices and denote by $\tilde{g}$ the entropy of an arbitrary greedy point $\tilde{x} \in STAB(G)$. Then*

$$\tilde{g} \leq \frac{1}{1-\delta}\left(H(G) + \lg \frac{1}{\delta}\right)$$

for all $\delta > 0$, and in particular

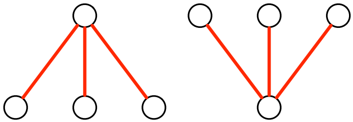
$$\tilde{g} \leq H(G) + \lg H(G) + O(1).$$

Proof idea. Dual fitting, using min-max relation

$$H(G) + H(\bar{G}) = \lg n$$

due to Csiszár, Körner, Lovász, Marton and Simonyi (90) $\square$

Colorings $\not\to$ Weak Order Extensions



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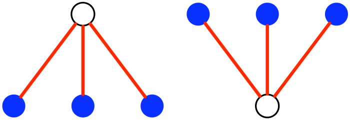

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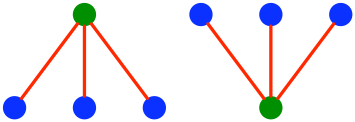

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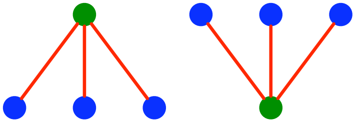


Weak order extensions of $P \to$ colorings of $G(P)$

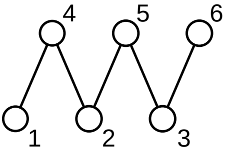
$\not\leftarrow$

$\Longrightarrow$ need to "uncross" our greedy colorings

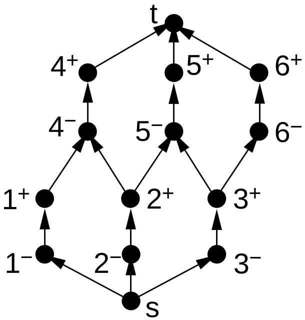
Uncrossing a Greedy Coloring

$D = D(P) :=$ auxiliary network with source s, sink t

$D = (N(D), A(D))$



$\rightsquigarrow$

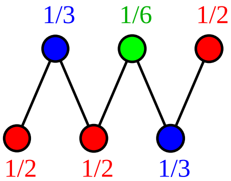


$$
\begin{array}{llrcll}
\text{(H-potential)} & \min & \multicolumn{4}{l}{-\dfrac{1}{n}\displaystyle\sum_{v \in V} \lg x_v} \\
 & \text{s.t.} & x_v & = & y_{v^+} - y_{v^-} & \forall v \in V \\
 & & y_a & \leqslant & y_b & \forall (a,b) \in A(D) \\
 & & y_s & = & 0 & \\
 & & y_t & = & 1 &
\end{array}
$$

Find potential $\tilde{y}$ for greedy point $\tilde{x}$ (by DP)

We get:

- collection of open intervals $\{\, (\tilde{y}_{v^-}, \tilde{y}_{v^+}) \,\}_{v \in V}$
- interval order I extending P, with $H(I)$ close to $H(P)$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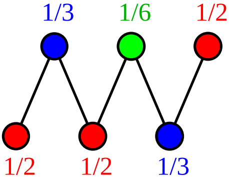
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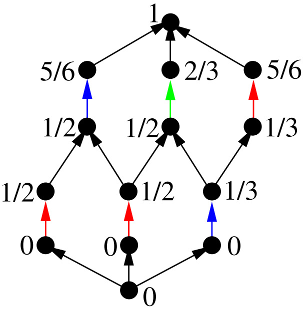
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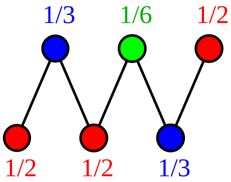
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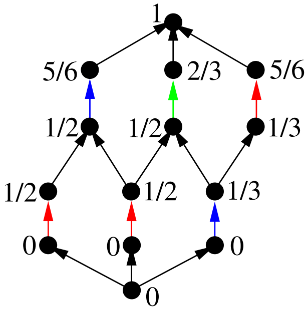
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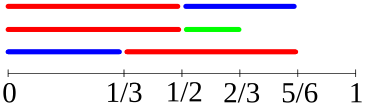
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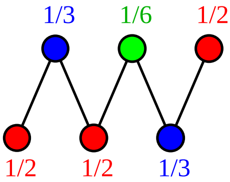
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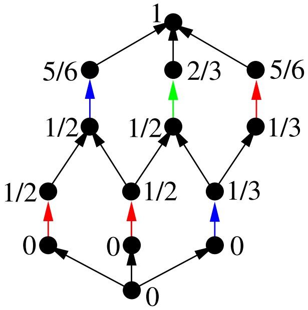
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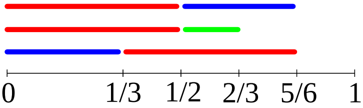
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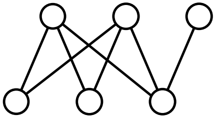

Main Steps of our Algorithm

1. $P \overset{greedy+DP}{\hookrightarrow} I$
2. $I \overset{greedy}{\hookrightarrow} W$
3. Use Multiple Selection algorithm of Kaligosi *et al.* on W

Theorem. *The algorithm above solves the POP problem, in $O(n^3)$ time, after performing at most*

$$LB + o(LB) + O(n)$$

comparisons

Further Result & Open Questions

Tightness result:

- Any algorithm reducing the POP problem to Multiple Selection can be forced to perform

$$LB + \Omega(n \lg \lg n)$$

  comparisons for some P with $H(P) \approx \frac{1}{2} \lg n$

Open questions:

- Is there an algorithm performing $LB + O(n)$ comparisons?
- What about Partial Order Production under Partial Information?

Thank You!

P.S.: The paper is available on ArXiv