Discrete Optimization with Ordering

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Discrete Optimization with ordering

Discrete optimization problems where feasible solutions are sequences of elements which are ordered with respect to a priority (hierarchy) function.

The cost of an element depends on its position on the sequence.

- Multiperiod problems
- Scheduling and sequencing problems
- …

- The simple ordering problem (SOP)
  - Ordered sequences
  - Some properties
  - The polyhedron of the SOP
- The simple ordering problem with cardinality constraint
- The simple ordering problem on an independence system
- The ordered median spanning tree problem

Ordered sequences

- Ground set: E= $\{e_1, e_2, \ldots, e_n\}$. $N = \{1, 2, \ldots, n\}$.
- c: E $\rightarrow \mathbb{R}$ order function s.t. $c_1 := c(e_1) \geq c(e_2) \geq \ldots \geq c(e_n) := c_n$
- Feasible Solutions: sequences with at most $p \leq n$ elements

  which are ordered wrt function c. $K = \{1, 2, \ldots p\}$.

  $[\, e_{j1}, e_{j2}, \ldots, e_{jr} \,]$, $r \leq p$, such that $j_i < j_{i+1}$

	e_{j1}		…		e_{j2}		…	e_{jr}	

- E_K: multiset with p copies of each element $e \in$ E.
  - $F \subseteq E_K$, $\quad F = \left\{ e_{j1}^{k_1}, e_{j2}^{k_2}, \ldots, e_{jr}^{k_r} \right\}$ with $k_1 \leq k_2 \leq \ldots \leq k_r$
  - $F \subseteq E_K$, ordered sequence $\Leftrightarrow k_i < k_{i+1}$ and $j_i < j_{i+1}$, $i = 1, \ldots, p-1$
- Additive objective function: The value of each element depends

  on its position in the sequence.

$$
\begin{aligned}
d:\ E_K &\longrightarrow \mathbb{R} \\
e_j^k &\longrightarrow d_j^k \\
F &\longrightarrow \sum_{e_j^k \in F} d_j^k
\end{aligned}
$$

Example

E = $\{e_1, e_2, e_3, e_4\}$ $(c_1 \geq c_2 \geq c_3 \geq c_4)$, $p = 3$.

$$d = \begin{pmatrix} 2 & 0 & 5 \\ 2 & 2 & 3 \\ 3 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

$F = \{e_2^{\,1}, e_2^{\,2}, e_3^{\,2}, e_1^{\,3}\}$ is not an ordered sequence

Ordered sequences:

$F_1 = \{e_2^{\,1}, e_3^{\,2}\}$, $F_2 = \{e_2^{\,2}\}$, $F_3 = \{e_1^{\,3}\}$

$d(F_1) = 2$, $d(F_2) = 2$, $d(F_3) = 5$

Some properties of ordered sequences

- I = (E_K,$\mathcal{F}$) is an Independence System,

  where $\mathcal{F}$ = {$F \subseteq E_K$: F is an ordered sequence}.

  $F \subseteq E_K$ ordered sequence $\Rightarrow$ S ordered sequence, for all $S \subseteq F$.

- $\ell(F)$ = max{ $|S|$: $S \subseteq F$, S is an ordered sequence}.

  F is an ordered sequence $\Leftrightarrow$ $\ell(F) = |F|$.

- I = (E_K,$\mathcal{F}$) is not a matroid

  For a given F, it is possible to find maximal ordered sequences S, $T \subseteq F$ such that $\ell(S) \neq \ell(T)$,

The Simple Ordering Problem (SOP):

Given E, c, p, d,

to find an ordered sequence of maximum total weight with respect to d.

$$d(F^*) := \max \quad d(F)$$
$$\text{s.t.} \quad |F| \le \ell(F), \qquad \text{for all } F \subseteq E_K$$

An optimal solution may have any number of elements in the range [0, p].

$$d = \begin{pmatrix} -2 & 0 & -5 \\ -2 & -2 & -3 \\ -3 & 0 & -1 \\ -1 & -1 & -2 \end{pmatrix} \qquad d = \begin{pmatrix} 2 & 0 & 15 \\ 2 & 2 & 3 \\ 3 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix} \qquad d = \begin{pmatrix} 2 & 0 & 5 \\ 2 & 2 & 3 \\ 3 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

The polyhedron of the SOP

$$x_{jk} = \begin{cases} 1 & \text{if element } e_j^k \text{ is selected} \\ 0 & \text{otherwise} \end{cases}$$

P_{SOP}= conv $\{x \in \{0, 1\}^{n \times p}: x(F) \leq \ell(F)$, for all $F \subseteq E_K\}$

- $\sum_{j \in N} x_{jk} \leq 1, \qquad k \in K$
- $\sum_{k \in K} x_{jk} \leq 1, \qquad j \in N$

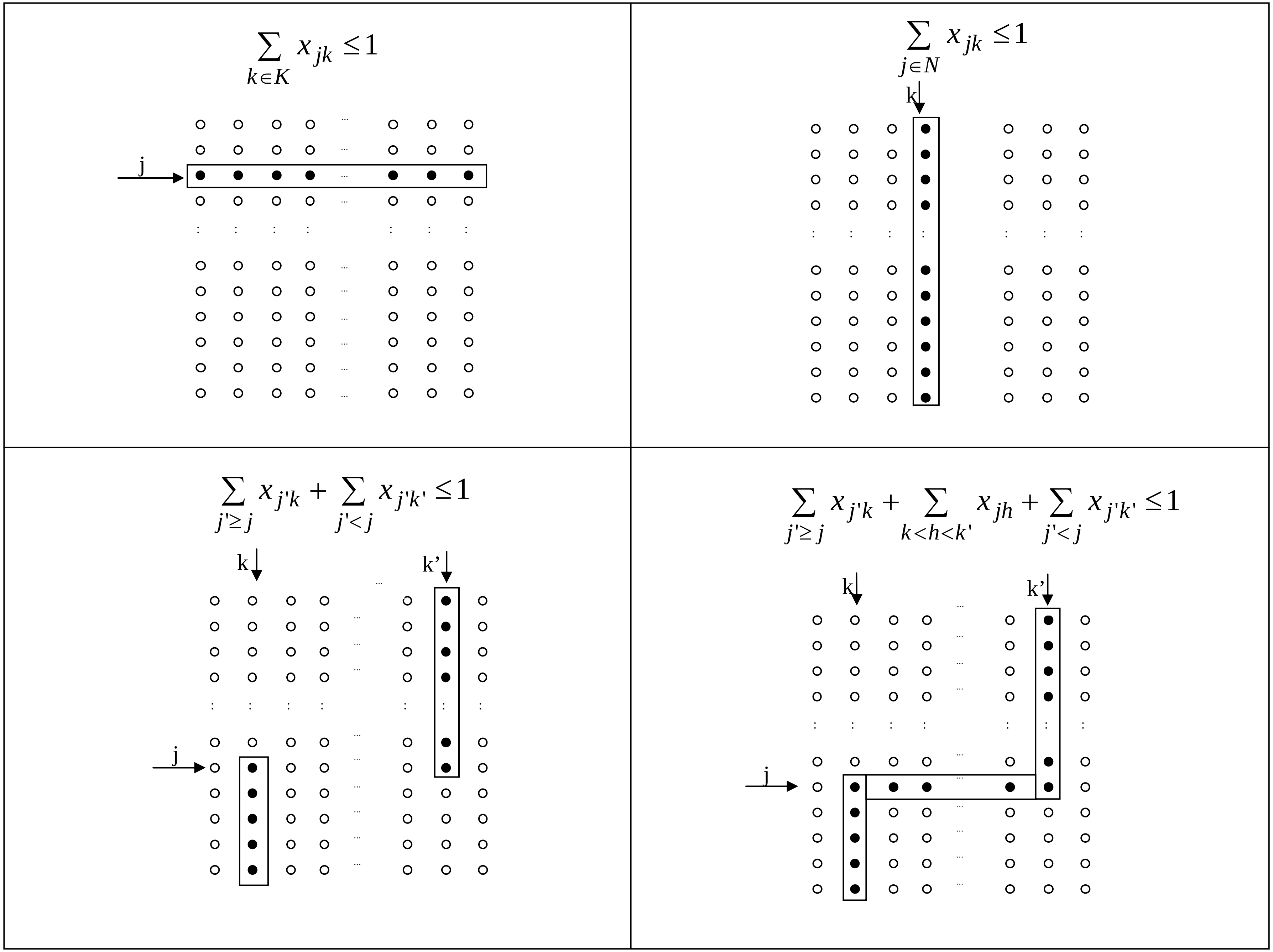
- $x_{jk} + \sum_{j' \leq j} x_{j'k'} \leq 1, \qquad \forall j \in N, \quad \forall k, k' \in K, k' > k$

$\Uparrow$

$$\sum_{j' \geq j} x_{j'k} + \sum_{j' \leq j} x_{j'k'} \leq 1, \qquad \forall j \in N, \quad \forall k, k' \in K, k' > k$$

$\Uparrow$

$$\sum_{j' \geq j} x_{j'k} + \sum_{k < h < k'} x_{jh} + \sum_{j' \leq j} x_{j'k'} \leq 1, \qquad \forall j \in N, \quad \forall k, k' \in K, k' > k$$


$\sum_{k \in K} x_{jk} \leq 1$
j
$\sum_{j \in N} x_{jk} \leq 1$
k
$\sum_{j' \geq j} x_{j'k} + \sum_{j' < j} x_{j'k'} \leq 1$
k
k'
j
$\sum_{j' \geq j} x_{j'k} + \sum_{k < h < k'} x_{jh} + \sum_{j' < j} x_{j'k'} \leq 1$
k
k'
j

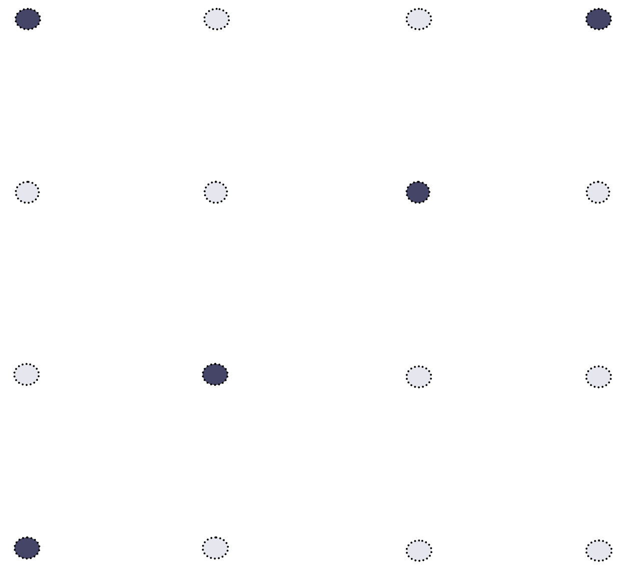
$$\sum_{j'\geq j} x_{j'k} + \sum_{k<h<k'} x_{jh} + \sum_{j'\leq j} x_{j'k'} \leq 1, \qquad \forall j \in N, \quad \forall k, k' \in K, k' > k$$

$$\sum_{k'\leq k} x_{jk'} + \sum_{j<h<j'} x_{hk} + \sum_{k'>k} x_{j'k'} \leq 1 \qquad \forall j, j' \in N, j' < j, \qquad \forall k \in K$$

Not enough

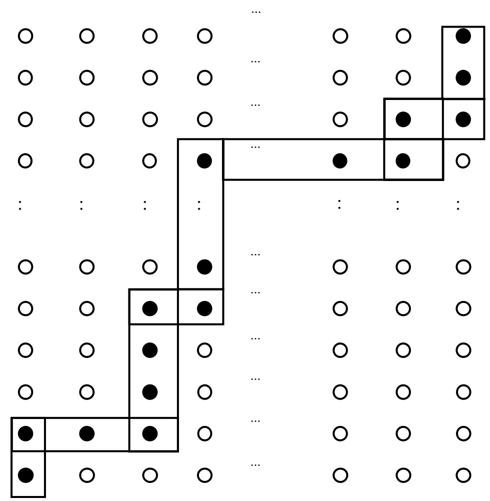
E = $\{e_1, e_2, e_3, e_4\}$ p=4,

$x_{11} = x_{41} = x_{32} = x_{23} = x_{14} = 1/3$



Staircase: $H \subseteq E_K$ s.t. e_j^k , $e_{j'}^{k'} \in H$ and $j' \leq j$, then $k \leq k'$.

Maximal staircase: staircase not contained in any other



Staircase inequality $\sum_{e_j^k \in H} x_{jk} \leq 1$

Valid inequality for the SOP that generalizes all the previous ones

H maximal staircase $\Rightarrow$ undominated inequality

Theorem:

$P_{SOP} = \{x \in [0, 1]^{n \times p} : x(H) \leq 1$, for all maximal staircase $H \subseteq E_K\}$

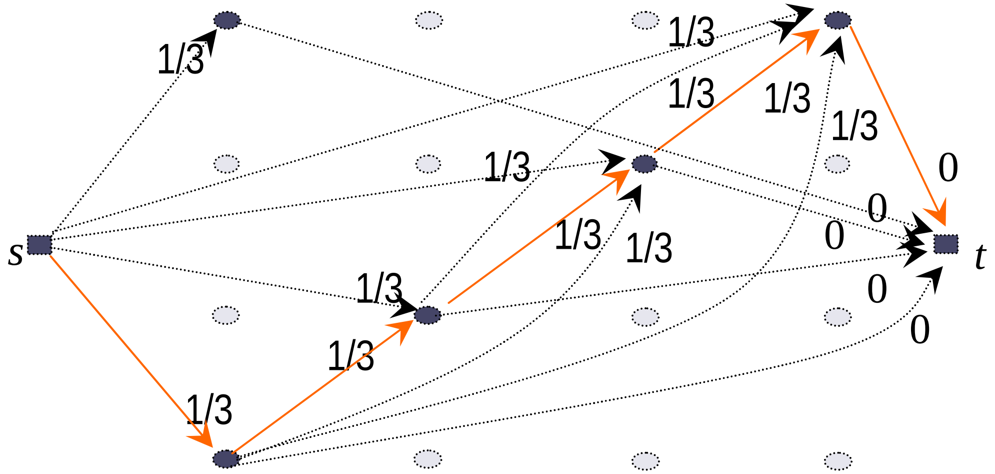
Separation of staircase inequalities

Proposition:

For a given $y \in \mathbb{R}^{n \times p}$ such that $0 \leq y_{jk} \leq 1$, for all $j \in N$, $k \in K$, the separation problem for staircase inequalities can be solved in polynomial time by finding an s−t−path of maximum cost in $N(y)$.

E = $\{e_1, e_2, e_3, e_4\}$ p=4,

$x_{11} = x_{41} = x_{32} = x_{23} = x_{14} = 1/3$



$N(y) = (V_y \cup \{s, t\}, A(y))$

- $V_y = \{v_j^k \in E_K: y_{jk} > 0\}$: support of y;
- s and t: fictitious source and sink.

$A(y)$ contains the following arcs:

- One arc (s, v_j^k) of cost y_{jk}, for each node $v_j^k \in V_y$.
- One arc $(v_j^k, v_{j'}^{k'})$ of cost $y_{j'k'}$ for each pair $v_j^k, v_{j'}^{k'} \in V_y$, with $j \geq j'$ and $k \leq k'$.
- One arc (v_j^k, t) of cost zero, for each node $v_j^k \in V_y$.

Theorem:

The SOP can be solved in polynomial time.

The simple ordering problem with cardinality constraint (SOPC)

Given E, c, p, d,

to find an ordered sequence that contains exactly one element of each $k\in K$, of maximal total weight with respect to d.

$$d(F^*) := \max \quad d(F)$$
$$\text{s.t.} \quad |F| \leq \ell(F), \qquad F \subseteq E_K.$$
$$|F| = p$$

$$\sum_{e_j^k \in H} x_{jk} \leq 1 \qquad \forall H \text{ maximal stair}$$

$$\sum_{j \in N} x_{jk} = 1, \qquad k \in K$$

$\Rightarrow$

$$\sum_{e_j^k \in H} x_{jk} \leq 1 \qquad \forall H \text{ maximal stair}$$

$$\sum_{k \in K} \sum_{j \in N} x_{jk} = p$$

Theorem:

$P_{SOPC} = \{x\in[0,\ 1]^{n\times p} : x(H) \leq 1$, for all maximal staircase $H \subseteq E_K$, $\sum_{k\in K}\sum_{j\in N} x_{jk} = p$ $\}$.

Theorem: The SOPC can be solved in polynomial time.

The simple ordering problem on an independence system (SOPI)

Given E, c, p, d + Independence System J =(E,$\mathcal{H}$)

I = ($\mathrm{E_K}$,$\mathcal{F}$) Independence System induced by order function c

J_K = ($\mathrm{E_K}$,$\mathcal{H}_K$) Independence System

$$\left\{e_{j_1}^{k_1}, e_{j_2}^{k_2}, \dots, e_{jr}^{k_r}\right\} \in \mathcal{H}_K \iff \left\{e_{j_1}, e_{j_2}, \dots, e_{jr}\right\} \in \mathcal{H}$$

to find an ordered sequence of $\mathcal{F}$ which is an independent set of $\mathcal{H}_K$ of maximum total weight with respect to d.

$\mathcal{L}(F)$ = max{$|S|$: $S \subseteq F$, $S \in \mathcal{F} \cap \mathcal{H}_K$}.

$d(F*)$:= max		$d(F)$	
	s.t.	$\|F\| \le \mathcal{L}(F)$,	$F \subseteq \mathrm{E_K}$.
		$\|F\|=p$	

$\mathrm{P_{SOPI}}$ = conv $\{x \in \{0, 1\}^{n\times p} : x(F) \le \mathcal{L}(F),\ F \subseteq \mathrm{E_K}\}$

$\mathrm{P_{SOPIC}}$ = conv $\{x \in \{0, 1\}^{n\times p} : x(F) \le \mathcal{L}(F),\ F \subseteq \mathrm{E_K},\ |F|=p\}$

Mathematical Programming Formulation of SOPI(C)

$$\begin{array}{lll}
Max \sum_{k\in K}\sum_{j\in N} d_j^k x_{jk} & & \\
s.t.\quad x(H)\le 1 & \text{for all maximal staricase } H\subset E_K & (1)\\
\quad x(F)\le r(F) & \text{for all } F\subseteq E_K & (2)\\
\quad \boxed{x(F)=p} & & (3)\\
\quad x_{jk}\in\{0,1\} & \text{for all } j\in N, k\in K &
\end{array}$$

$r(\cdot)$: rank function on J_K

$\{x \in \{0, 1\}^{n\times p}$: x satisfies (1) and (2)$\}$ has fraccional vertices

E = $\{e_1, e_2, e_3\}$ edges of K_3, p=3 and $\mathcal{H}$ given by forests in K_3

$$d=\begin{pmatrix}1 & 0 & 1.5\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

$x_{11}= x_{22}= x_{13}= x_{33}$=1/2 z*=2.25

Some properties of $P_{SOPI} = conv\,\{x \in \{0, 1\}^{n \times p} : x$ satisfies (1) and (2)$\}$

- Inequalities $x(F) \leq r(F)$ need not be facets of P_{SOPI}

E = $\{e_1, e_2, e_3\}$ edges of K_3, p=3 and $\mathcal{H}$ given by forests in K_3

$$F=\{e_2^1,\ e_2^2,\ e_3^3\}, \quad r(F) = 2$$

$$x_{11} + x_{22} + x_{33} \leq 2$$

is dominated by

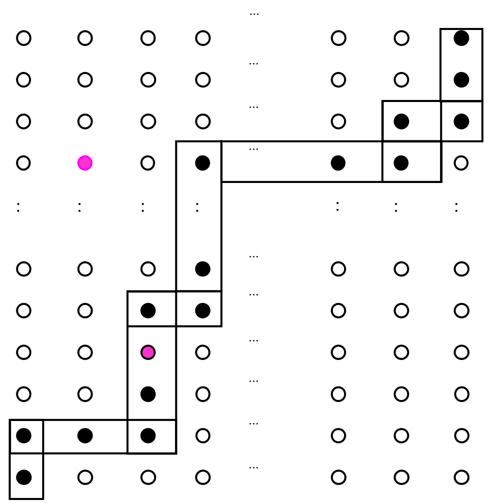
$$x_{11} + x_{22} + x_{32} + x_{33} \leq 2$$

More properties of P_{SOPI} = *conv* $\{x \in \{0, 1\}^{n \times p} : x$ satisfies (1) and (2)$\}$

Proposition: Let $H \subset E_K$ be a maximal staircase such that

for all $e_j^k \in E_K \setminus H$ $\exists$ $e_{j'}^{k'} \in H$ s.t. $\{e_j^k, e_{j'}^{k'}\} \in \mathcal{F} \cap \mathcal{H}_K$.

Then, $x(H) \leq 1$ is a facet of P_{SOPI}.



More properties of P_{SOPI} = *conv* $\{x \in \{0, 1\}^{n \times p} : x$ satisfies (1) and (2)$\}$

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Then, $x(H) \leq 1$ is a facet of P_{SOPI}.

Corollary: If $\{e_j, e_{j'}\} \in \mathcal{H}$ for all $j, j' \in N$, then for all maximal staircase H, $x(H) \leq 1$ is a facet of P_{SOPI}.

Conditions hold:

- $J = (E, \mathcal{H})$: forests in graph (V, E)
- $J = (E, \mathcal{H})$: sets of l.i. columns of an $m \times n$ matrix A such that not two columns are l.d.
- $J = (E, \mathcal{H})$: independence system induced by a knapsack type constraint with coefficients vector $a \in \mathbb{R}^n$ and right hand side a_0 such that $a_j + a_{j'} \leq a_0$ for all $j, j', j \neq j'$.

Conditions do not hold:

- $J = (E, \mathcal{H})$: matchings in graph (V,E)

Ordered median objective function

Given	Ordered median of F
E, p, K c: E $\to \mathbb{R}$, λ: $\{1, 2, \ldots, p\} \longrightarrow \mathbb{R}^+$ $F \subseteq$ E, $\|F\| = p$,	$$om(F) = \sum_{k \in K} \lambda_k c_{e_{\pi_F(k)}}$$ π_F : K $\longrightarrow$ K t.q. $c_{\pi_{F(1)}} \geq c_{\pi_{F(2)}} \geq \cdots \geq c_{\pi_{F(p)}}$.

If	**λ=(1, 1 … , 1, …, 1)**	**Sum**	**(median)**
If	**λ=(1, 0 … , 0, …, 0)**	**Maximum**	**(center)**
If	**λ=(1, 1, … ,1, …, 0)**	**k-center**	
If	**λ=(1, α, …, α, …, α)**	**Cent dian**	

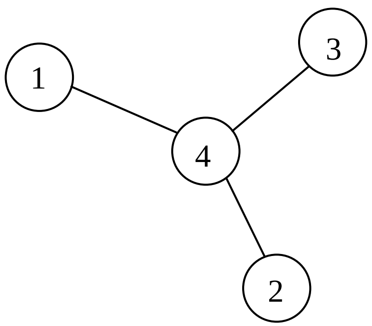
Nickel, S., and Puerto, J. (2005). Location Theory. A Unified approach. Springer.

The ordered median spanning tree problem

Given $G=(V, E)$, $c = (c_1, \dots c_n)$, $c_1 \geq \dots \geq c_n$, $\lambda = (\lambda_1, \dots, \lambda_p) \geq 0$, the Ordered Median Spanning Tree Problem (OMSTP) consists of finding an ordered sequence $\{e^{k1}_{j1}, \dots, e^{kp}_{jp}\}$ such that $T = \{e_{j1}, \dots, e_{jp}\}$ is an spanning tree of G, that maximizes the value of $\sum_{k \in K} \lambda_k c_{jk}$.

	2	3	4
1	4	3	5
2		6	8
3			7

$\lambda_1=10$, $\lambda_2=1$, $\lambda_3=5$



$c_{24} \geq c_{34} \geq c_{14}$

om(T)= 10 x 8 + 1 x 7 + 5 x 5

The OMSTP

$$
\begin{aligned}
& Max \sum_{k \in K} \sum_{j \in N} \lambda_k c_j x_{jk} \\
& s.t. \quad x(H) \le 1 && \text{for all maximal staricase } H \subset E_K \\
& \qquad x(F) \le r(F) && \text{for all } F \subseteq E_K \\
& \qquad x(F) = p \\
& \qquad x_{jk} \in \{0,1\} && \text{for all } j \in N, k \in K
\end{aligned}
$$

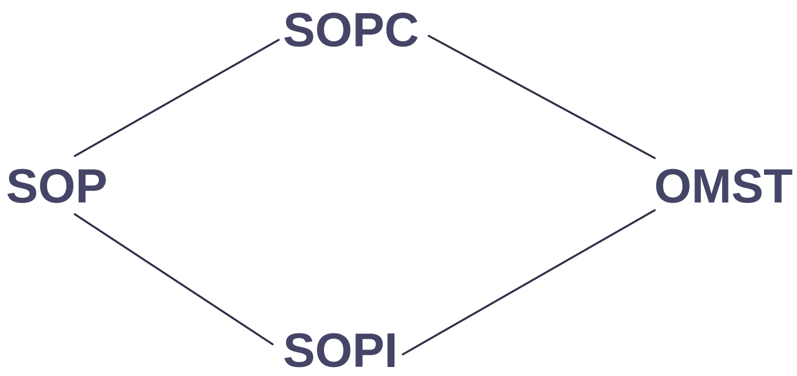
The OMSTP is a SOPIC with $d_j^k = \lambda_k c_j$ and $J = (E, \mathcal{H})$ given by forests in G.

Proposition:

The greedy algorithm yields an optimal solution to the OMSTP

Concluding remarks

- Discrete optimization problems with ordering



- **The paper can be found at** **http://www-eio.upc.es/~elena/doo.pdf**

Future work

- Characterization of P_{SOPI}
- Study other (easy) problems with order
- Study other (less easy) problems with order
- …

Thank you for your attention!