Extended formulations for Packing and Partitioning Orbitopes

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joint work with

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Outline

- What Packing and Partitioning Orbitopes are, and what for ?
- P & P orbitopes in the original and in extended spaces.
- New proof of the complete description in the original space.

Packing and partitioning Orbitopes

Consider all $0-1$ matrices with p rows and q columns that have

(1) *exactly* one non-zero entry per row;

(2) columns in lexicographical non-increasing order;

Their convex hull is the *partitioning orbitope* $\mathrm{O}^{=}_{p,q}$.

0	0
0	0
1	0
0	1
0	0
1	0

1					
1	0				
0	1	0			
0	1	0	0		
1	0	0	0	0	
0	0	1	0	0	0
0	0	0	1	0	0
0	1	0	0	0	0

Packing and partitioning Orbitopes

Consider all $0-1$ matrices with p rows and q columns that have

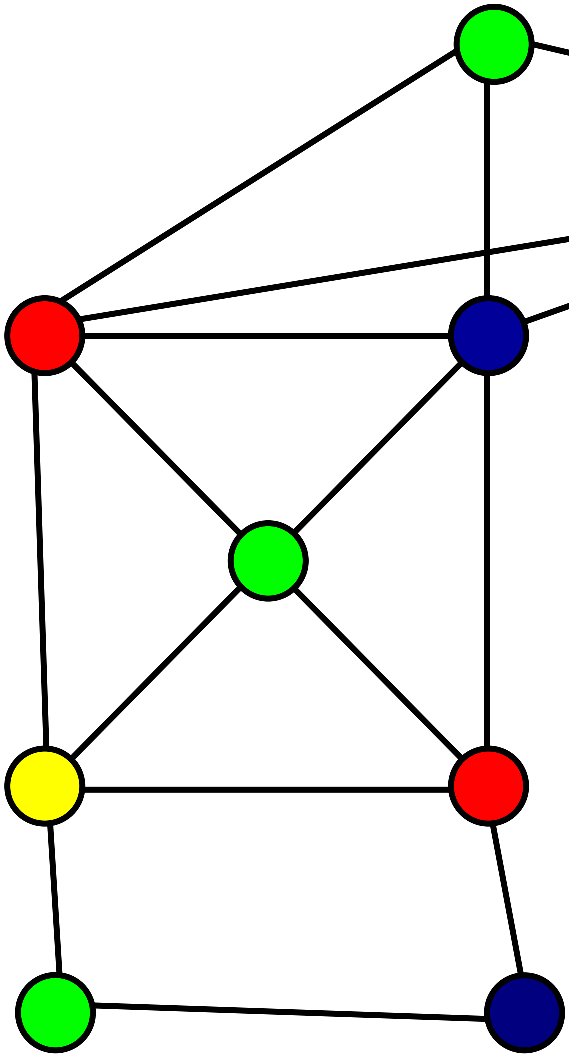
(1) *at most* one non-zero entry per row;

(2) columns in lexicographical non-increasing order;

Their convex hull is the *packing orbitope* $\mathrm{O}^{\leq}_{p,q}$.

$$\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{array} \qquad \begin{array}{|c|c|c|c|c|c|} \hline 0 & & & & & \\ \hline 1 & 0 & & & & \\ \hline 0 & 1 & 0 & & & \\ \hline 0 & 0 & 0 & 0 & & \\ \hline 1 & 0 & 0 & 0 & 0 & \\ \hline 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 \\ \hline \end{array}$$

Breaking symmetry in Graph Coloring



$$
\begin{array}{lll}
\min & \sum_{j=1}^{|S|} y_j & \\
\text{s.t.} & x_{ij} + x_{kj} \leq y_j, & \{i,k\} \in E,\ j \in S \\
& \sum_{j=1}^{|S|} x_{ij} = 1, & i \in V \\
& x_{ij} \in \{0,1\} & i \in V, j \in S \\
& y_j \in \{0,1\} & j \in S
\end{array}
$$

Breaking symmetry using Orbitopes

- Any column-permutation of $\begin{pmatrix} y^T \\ X \end{pmatrix}$ leads to an "equivalent solution"
- Thus, the solution space is unnecessarily large
- Idea: select one representative from each set of equivalent solutions.
- $\mathrm{IP} \cap \mathrm{O}^{=}_{p,q}$ as a symmetry-free formulation
- Re-use $\mathrm{O}^{=}_{p,q}$ for different problems
- Change the number of 1s per row, "new" orbitopes!

Some facts on P&P Orbitopes

Introduced by Kaibel and Pfetsch [*Math.Progr.*08], who gave:

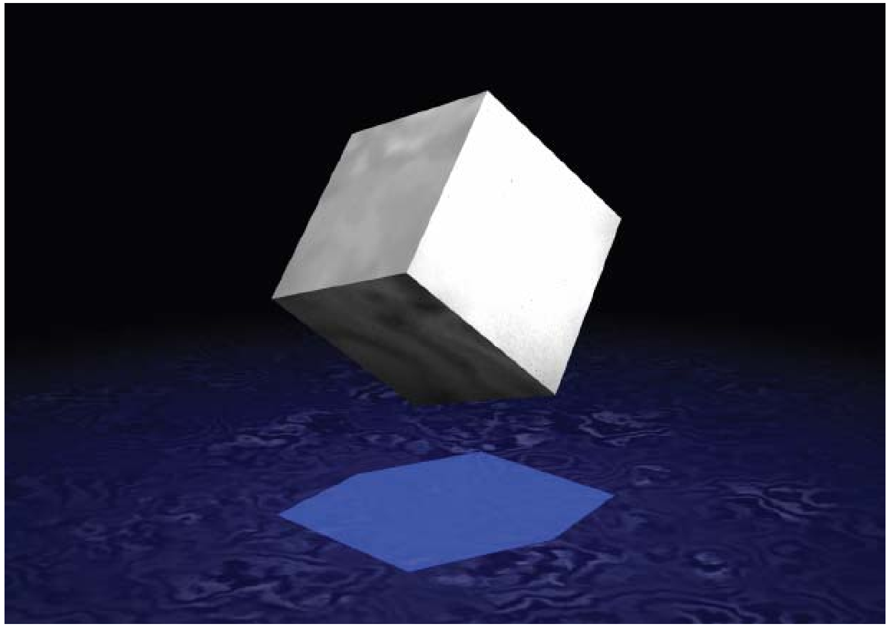
- an $O(p^2q)$ algorithm for optimizing a linear function over packing and partitioning orbitopes
- a complete ad irredundant description in the original space of $\mathrm{O}^{=}_{p,q}$ (resp. $\mathrm{O}^{\leq}_{p,q}$) (**SCI THR.**):
  - $x \geq 0$
  - $x(row_i) = 1$ (resp. $x(row_i) \leq 1$)
  - exponentially many shifted column inequalities

Extended formulations

$$P = \{x \in \mathbb{R}^n | Ax \leq b\}$$

$$Q = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m | Bx + Cy \leq d\}$$

$$Proj_x(Q) = P$$

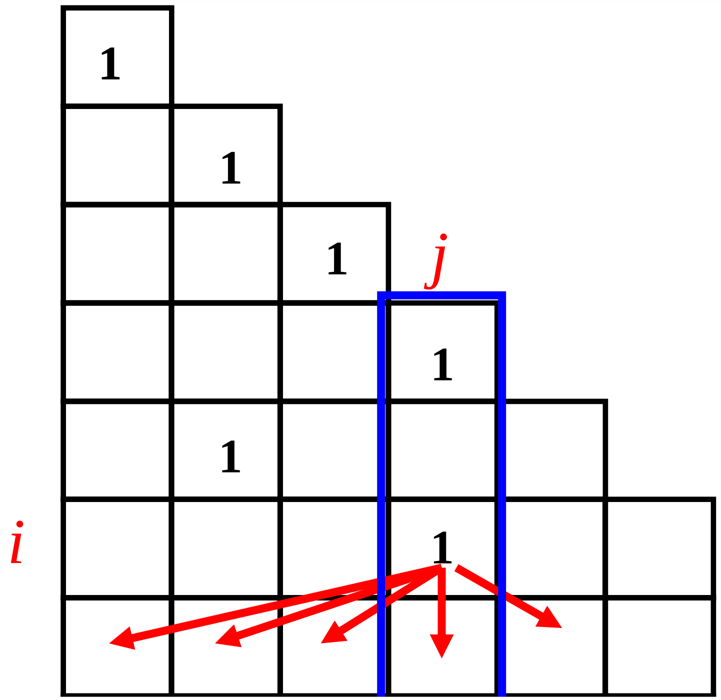


Lifting partitioning orbitopes (1)

Main idea:

Associate each integral $x \in \mathrm{O}^{=}_{p,q}$ with an $s-t$ path on a digraph

Kipp Martin, Rardin, Campbell [$Oper.Res.$90]



$j(i)$ =maximum column with a non-zero entry, up to row i

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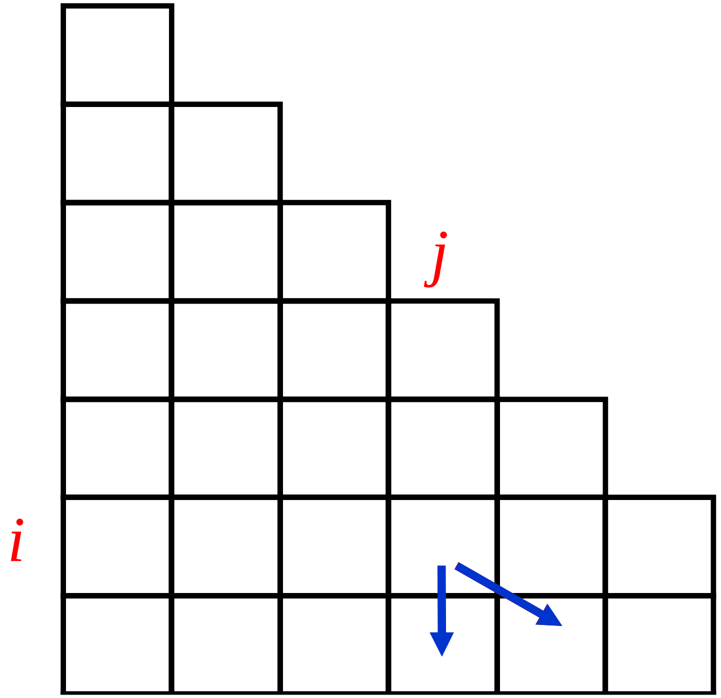

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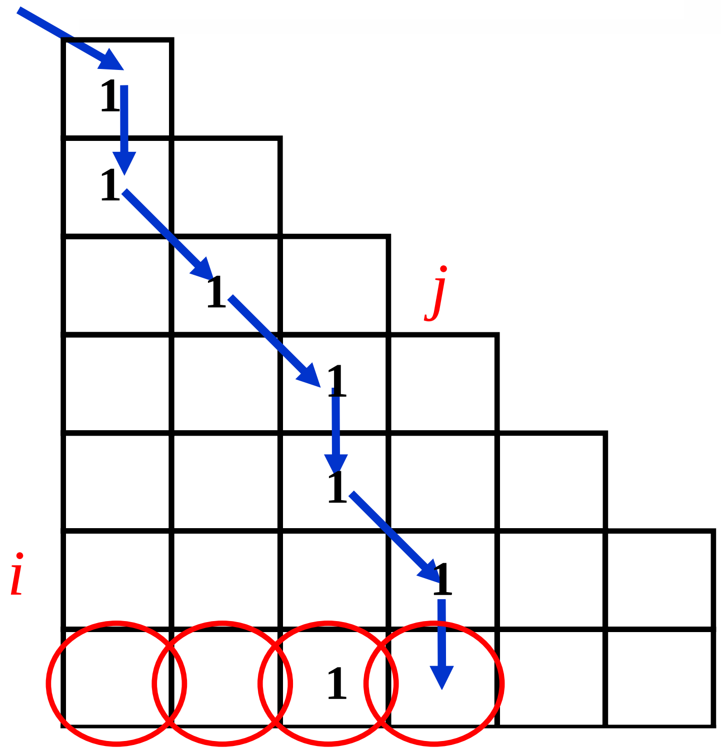

$j(i)$ =maximum column with a non-zero entry, up to row i

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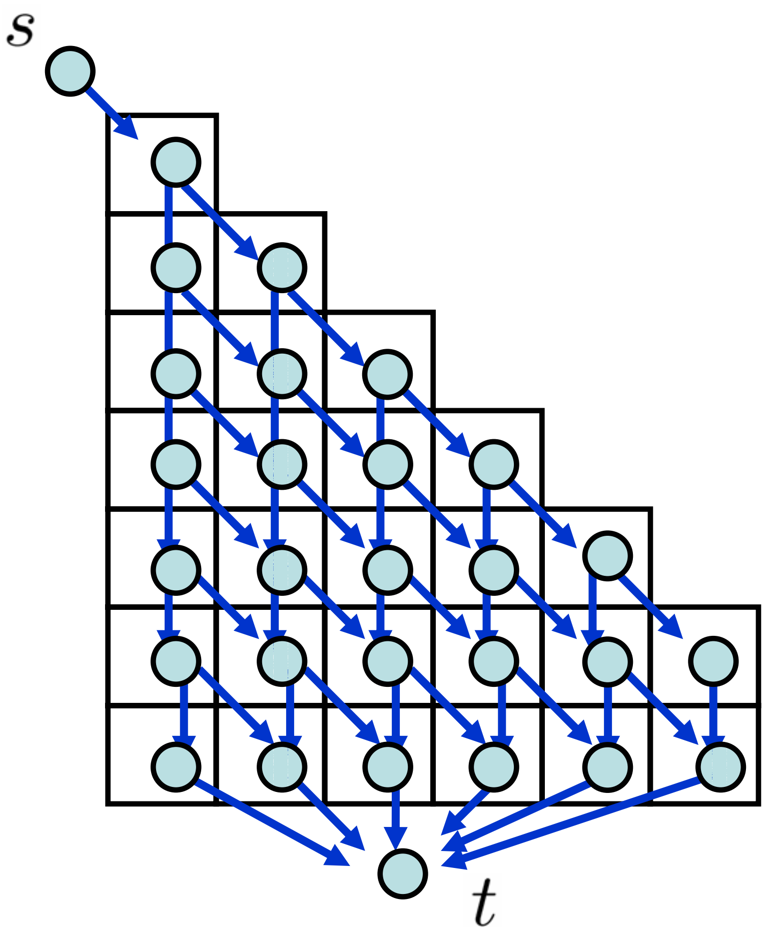
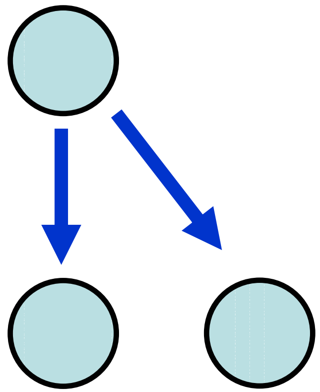
A single path in general corresponds to more vertices of $\mathrm{O}^{=}_{p,q}$

Lifting partitioning orbitopes (2)



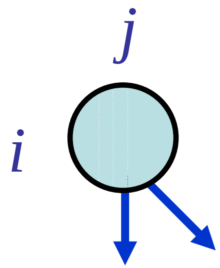
$$\max cx \text{ s.t. } x \in \mathrm{O}^{=}_{p,q}$$



$$c^{\star}_{(i,j)\searrow} = c_{i+1,j+1}$$
$$c^{\star}_{(i,j)\downarrow} = \max_{\ell \le j} c_{i+1,\ell}$$

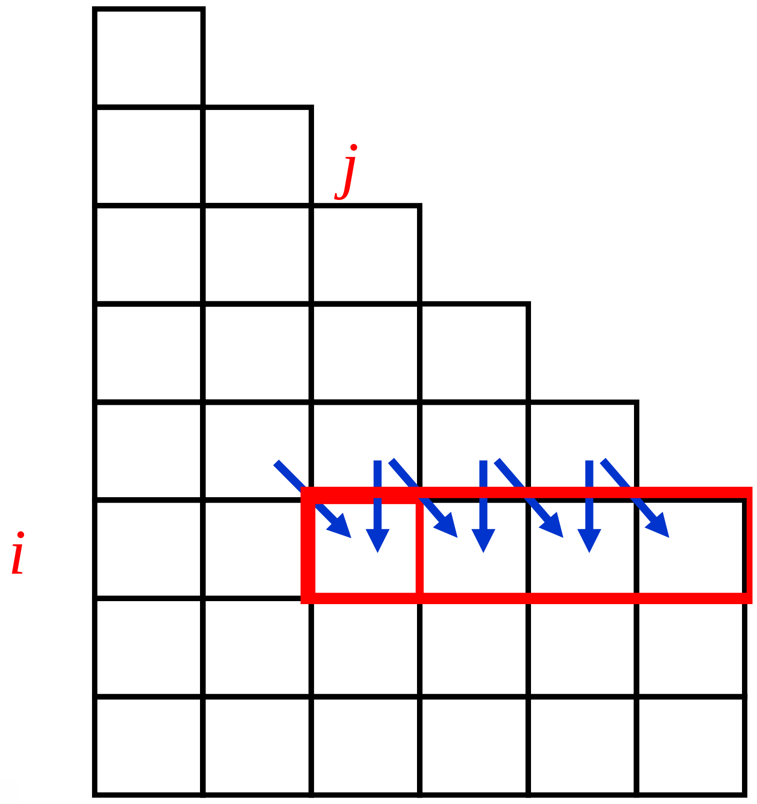
Thr.(F. & Kaibel 08): The problem $\max cx$ s.t. $x \in \mathrm{O}^{=}_{p,q}$ can be solved in time $O(pq)$.

The extended formulation



$$y_{(i,j)^{\downarrow}}$$

$$y_{(i,j)^{\searrow}}$$



$$P_{p,q} = \{(x,y) \,|$$

$$y \in F_{p,q}$$

$$x(\mathrm{row}_i) = 1$$

$$y_{(i-1,j-1)^{\searrow}} \le x_{i,j}$$

$$\sum_{m=j}^{i} x_{i,m} \le y_{(i-1,j-1)^{\searrow}} + \sum_{m=j}^{i-1} \left(y_{(i-1,m)^{\searrow}} + y_{(i-1,m)^{\downarrow}}\right) \}$$

Thr.(F. & Kaibel 08): $P_{p,q}$ is an extended formulation for $\mathrm{O}^{=}_{p,q}$.

Integrality of $\mathcal{P}$ – sketch of the proof

Suppose you want to solve $\quad \max cx$ s.t. $x \in \mathrm{O}_{p,q}^{=}$

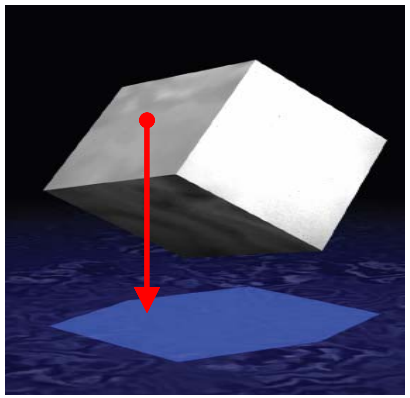
Define a new cost vector $c^\star$

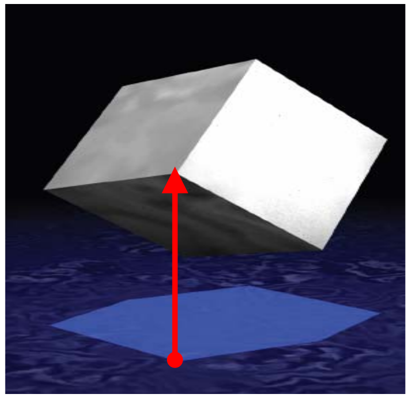
Claim1

$$< c^\star, (0,y) > \quad \geq \quad < c, (x,y) > \quad \forall (x,y) \in P_{p,q}$$

Claim2

For each integral $y \in F_{p,q}$ exists integral x with
$(x,y) \in P_{p,q}$ and $< c, (x,y) > \; = \; < c^\star, (0,y) >$.

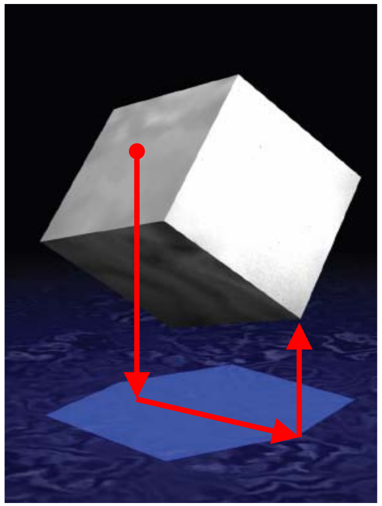




Integrality of $\mathcal{P}$ – sketch of the proof

Thus, for each $(\bar{x}, \bar{y}) \in P_{p,q}$:

$$
\begin{aligned}
< c, (\bar{x}, \bar{y}) > \quad &\leq \quad < c^\star, (0, \bar{y}) > \leq \max_{y \in F_{p,q}} < c^\star, (0, y) > \\
&= \quad < c^\star, (0, \tilde{y}) > \quad \text{for some integral } \tilde{y} \\
&= \quad < c, (\tilde{x}, \tilde{y}) > \quad \text{for } \tilde{x} \text{ integral,} (\tilde{x}, \tilde{y}) \in P_{p,q} \quad \square
\end{aligned}
$$



More on the Extended Formulation

- Once you prove integrality, projecting is easy
- after suitable transformations, we end up with a "very compact" formulation with less than $2pq$ variables, $4pq$ constraints and $10pq$ total nonzero elements.
- (almost) identical results hold for $\mathrm{O}_{p,q}^{\leq}$ (actually, all the work is done for $\mathrm{O}_{p,q}^{\leq}$)

Re-proving the SCI - theorem

The SCI-theorem is not necessary in our proofs

Use the extended formulation to obtain a new proof of the complete description in the original space. Why ?

- Find a shorter proof
- Get new insight on the problem

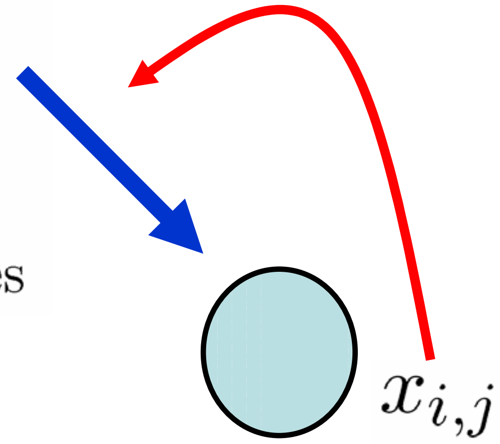
Let $Q_{p,q}$ be the SCI-polytope. Since we already proved $\mathrm{Proj}_x(P_{p,q}) = \mathrm{O}^{=}_{p,q}$ we are left to prove

- $\mathrm{O}^{=}_{p,q} \subseteq Q_{p,q}$;
- $Q_{p,q} \subseteq \mathrm{Proj}_x(P_{p,q})$

Re-proving the SCI-theorem (1)

(**SCI THR.**): $\mathrm{O}_{p,q}^{=}$ is completely described by:

- $x \geq 0$
- $x(row_i) = 1$
- exponentially many shifted column inequalities



New proof:

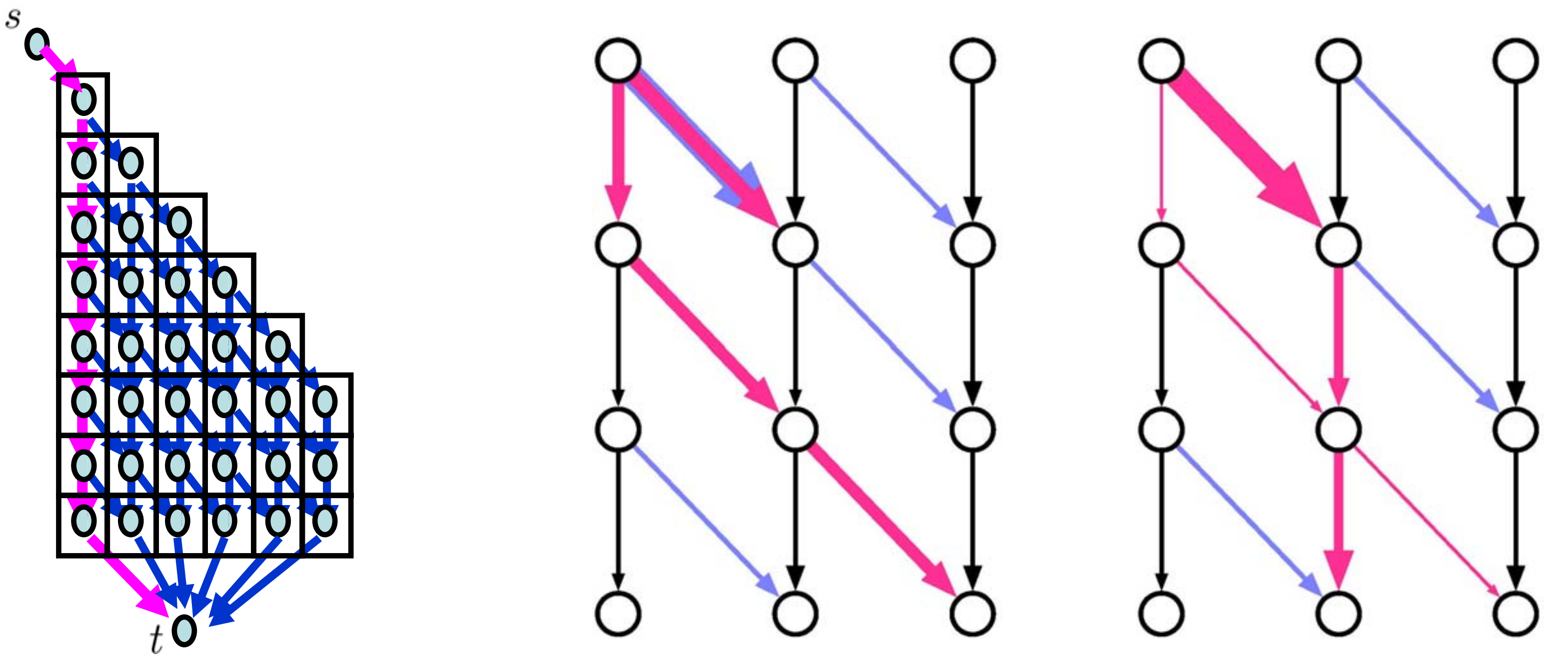
(1) Start from a point x s.t. $x(\mathrm{row}_i) = 1$ for each i and $x \geq 0$

(2) consider a network on digraph D with

- capacity $+\infty$ on vertical arcs
- capacity $x_{i,j}$ on the diagonal arc entering node (i, j)

(3) Construct the *rightmost* flow $y = \Pi(x)$

Constructing the rightmost flow

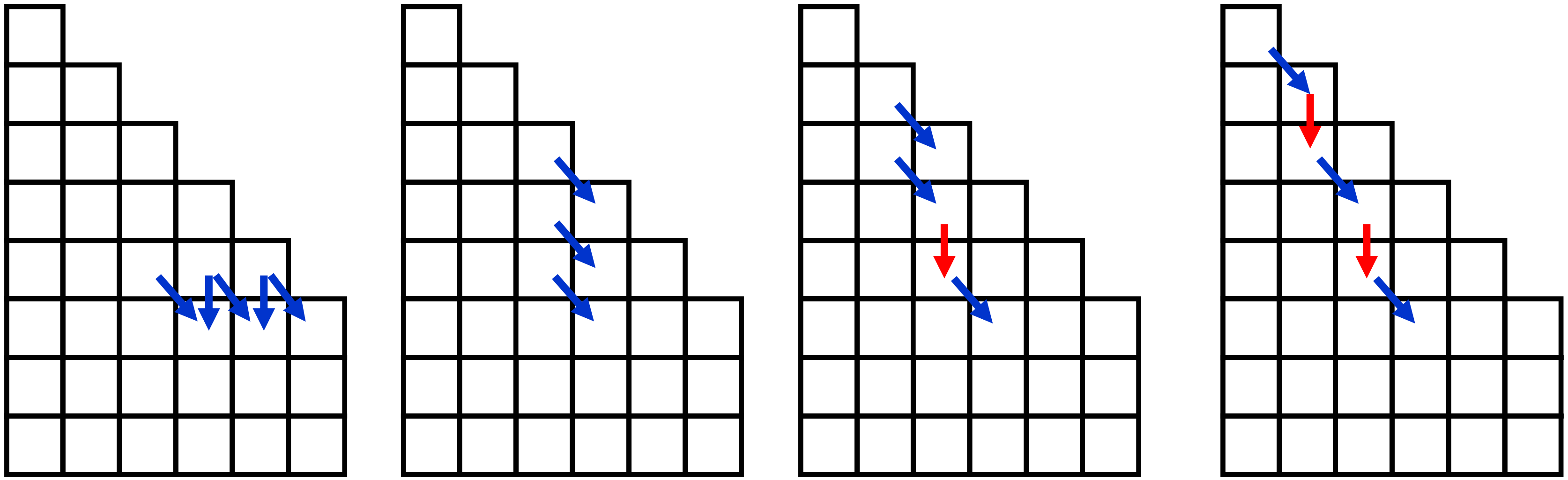


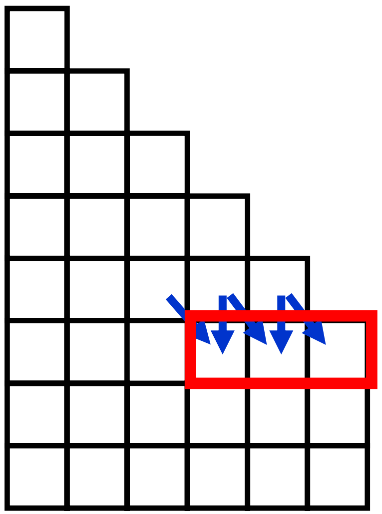
(4) We shall prove that $(x, y = \Pi(x)) \in P_{p,q}$ if the SCI are valid

(5) $y_{(i-1,j-1)^{\searrow}} \leq x_{i,j}$ come for free

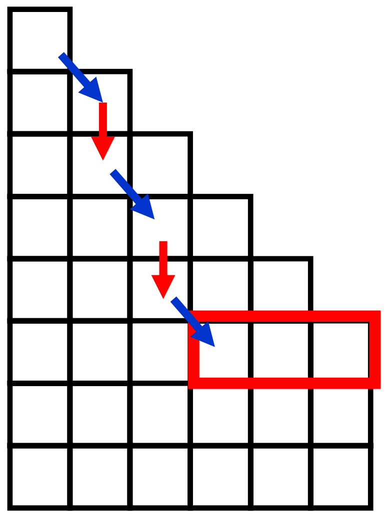
(6) while trying to prove UB inequalities, SCI pops up!

Equivalent flows





≥ 0



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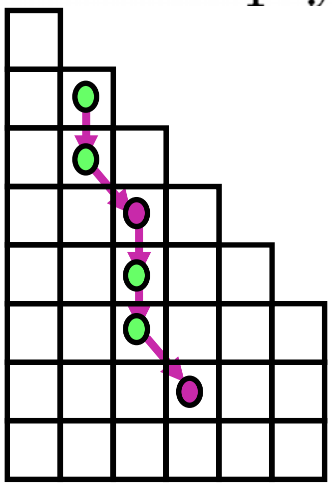
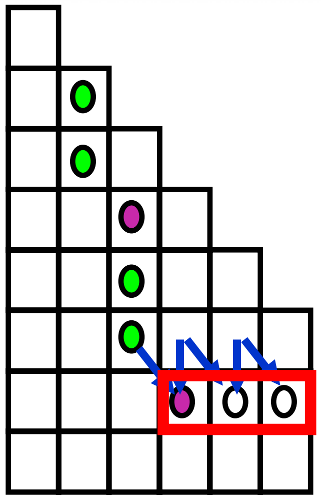
Re-proving the SCI-theorem (3)

(7) Given (i, j), build a backward leftmost flow

(8) SCIs imply UB on the x variables





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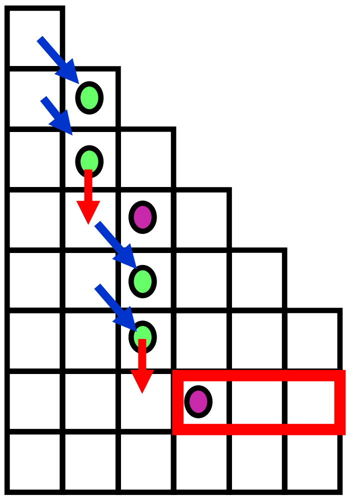

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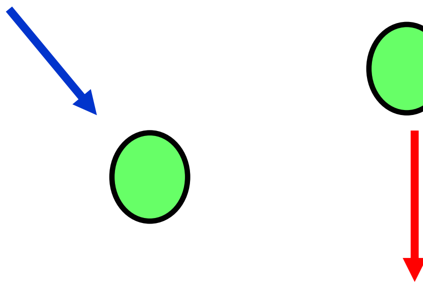

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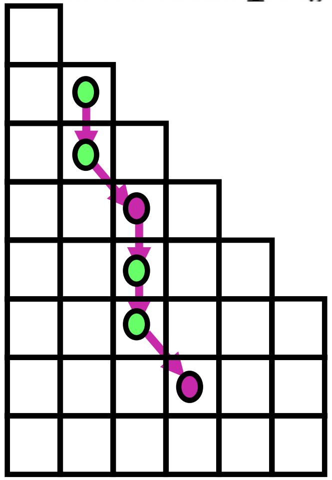
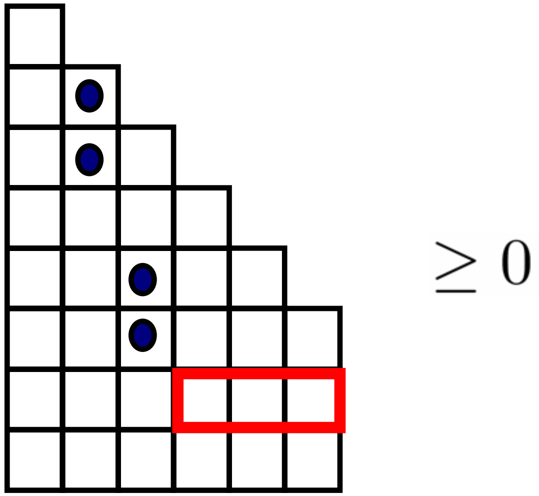
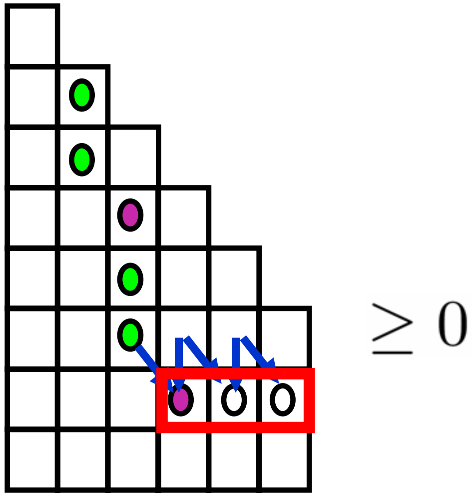
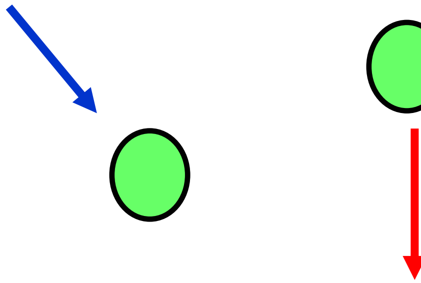






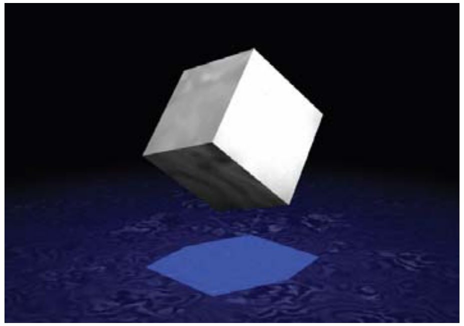


Shifted column inequality!

Summary and conclusions

Extended formulations for Packing and Partitioning orbitopes...

- (very) compactly describe the polytopes;
- let us optimize faster;
- give more insight;
- shorten proofs;



Thank you!

Paper available at:

http://www.math.uni-magdeburg.de/~kaibel/