A class of matrices with the Edmonds-Johnson property

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Definition

The *strong Chvátal rank* of a rational matrix A is the smallest number t such that the polyhedron defined by the system $b \leq Ax \leq c, l \leq x \leq u$ has Chvátal rank at most t for all integral vectors b, c, l, u.

Matrices with strong Chvátal rank 0 are exactly the totally unimodular matrices.

Definition

Matrices with strong Chvátal rank at most 1 are said to have the *Edmonds-Johnson property*.

There are two main known classes of matrices with the Edmonds-Johnson property:

Theorem (Edmonds and Johnson, '73)

If $A = (\alpha_{ij})$ is an integral matrix such that $\sum_i |\alpha_{ij}| \leq 2$ for each column index j, then A has the Edmonds-Johnson property.

Theorem (Gerards and Schrijver, '86)

An integral matrix (α_{ij}) *that satisfies* $\sum_j |\alpha_{ij}| \leq 2$ *for each row index* i, *has the Edmonds-Johnson property if and only if it cannot be transformed to* $M(K_4)$ *by a series of the following operations:*

- *deleting or permuting rows or columns, or multiplying them by* -1,
- *replacing matrix* $\begin{pmatrix} 1 & g \\ f & D \end{pmatrix}$ *by the matrix* $D - fg$.

$$M(K_4) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

The matrices in these two classes are *totally half-modular*, i.e. integral matrices such that for each nonsingular square submatrix B, $2B^{-1}$ is integral.

If a matrix A is totally half-modular, then the irredundant Chvátal-Gomory inequalities for the system $b \leq Ax \leq c, l \leq x \leq u$ are obtained with Chvátal-Gomory multipliers that have entries in $\{0, \frac{1}{2}\}$, for all integral vectors b, c, l, u.

Observation

The class of totally half-modular matrices with the Edmonds-Johnson property is closed under the following operations:

(i) *deleting or permuting rows or columns, or multiplying them by* -1,

(ii) *dividing by* 2 *an even row,*

(iii) *pivoting on a* 1 *entry, i.e. replacing matrix* $\begin{pmatrix} 1 & g \\ f & D \end{pmatrix}$ *by the matrix* $\begin{pmatrix} -1 & g \\ f & D - fg \end{pmatrix}$.

Definition

We say that a matrix B is a *minor* of A if it arises from A by a series of operations (i)-(iii).

Conjecture (Gerards and Schrijver)

A totally half-modular matrix has the Edmonds-Johnson property, if and only if it has no minor equal to A_4 or A_3.

$$A_4 = \begin{pmatrix} 2 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \\ 2 & 0 & 1 & 1 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 2 & 2 \\ 1 & 0 & 2 \end{pmatrix}$$

The two cited classes of matrices with the Edmonds-Johnson property are particular cases of Gerards and Schrijver's conjecture.

Theorem

A totally half-modular matrix obtained from a $\{0, \pm 1\}$-matrix with at most two nonzero entries per column, by multiplying by 2 some columns, has the Edmonds-Johnson property if and only if it does not contain A_3 as a minor.

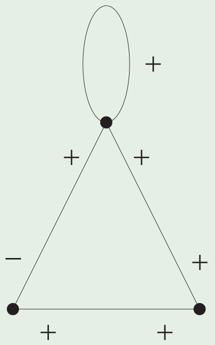
- Let A be obtained from a totally unimodular matrix with two nonzero elements per column, by multiplying by 2 some columns, and let b be an integral vector. Deciding if a system $Ax = b, x \geq 0$ has an integral solution is NP-complete. (Conforti, Di Summa, Eisenbrand, Wolsey, '08).
- This is a nontrivial class of matrices where appears the matrix A_3.
- Our result reduces to the one of Edmonds and Johnson when $\sum_i |\alpha_{ij}| \leq 2$ for each column index j.

Definition

A *bidirected graph* G is a triple (V, E, σ) where:

- (V, E) is an undirected graph,
- σ is a *signing* of (V, E), i.e. a map that assigns to each $e \in E, v \in e$ a *sign* $\sigma_{v,e} \in \{+1, -1\}$.
  For convenience, we define $\sigma_{v,e} = 0$ if $v \notin e$.

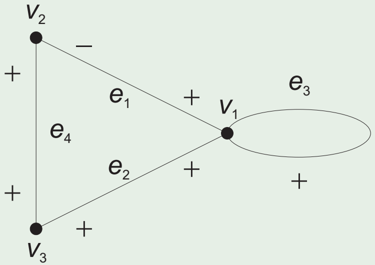
Example



Definition

The *sign matrix* of a bidirected graph G is the $|V| \times |E|$ matrix $\Sigma(G) = (\sigma_{v,e})$.

Example

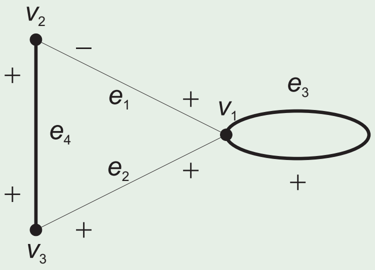


$$\Sigma(G) = \begin{pmatrix} 1 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

Definition

Given a bidirected graph G and a subset F of its edges, we denote with $A(G,F)$ the matrix obtained from $\Sigma(G)$ by multiplying by 2 the columns corresponding to the edges in F.

Example (G_4)



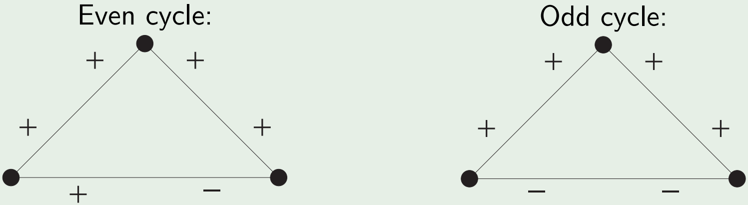
$$A(G,F) = \begin{pmatrix} 1 & 1 & 2 & 0 \\ -1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \end{pmatrix}$$

(the boldfaced edges represent the edges in F)

Definition

A cycle C is *even* if the number of edges in C with the same sign in its endnodes is even, and is *odd* otherwise.

Example

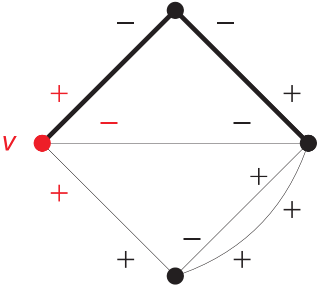


Observation

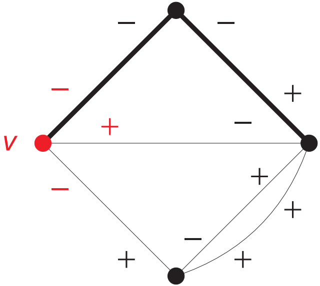
$A(G, F)$ is totally half-modular if and only if (G, F) satisfies the Cycles condition: the cycles of G containing edges in F are even.

Switch sign on a node v:

(G,F)

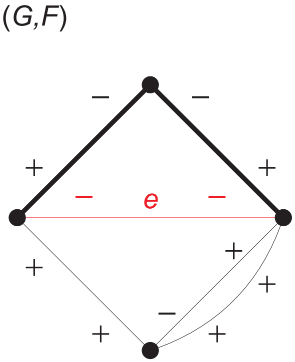


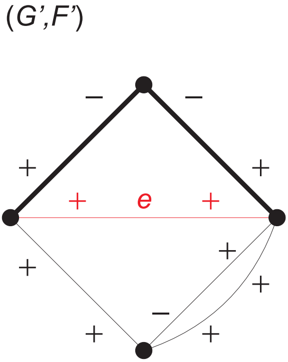
(G',F')



Corresponds to multiplying by -1 the row of $A(G,F)$ corresponding to v.

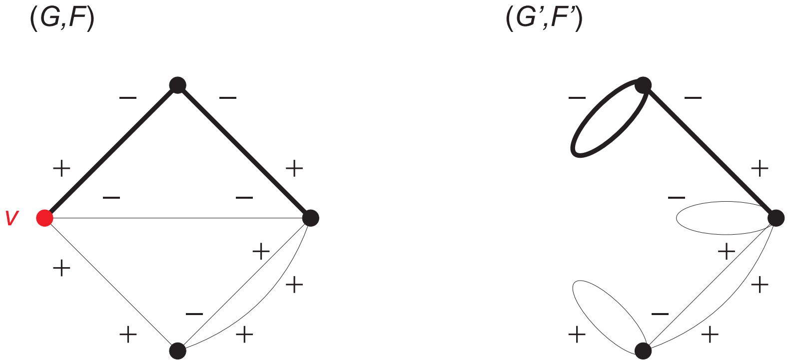
Switch sign on an edge e:





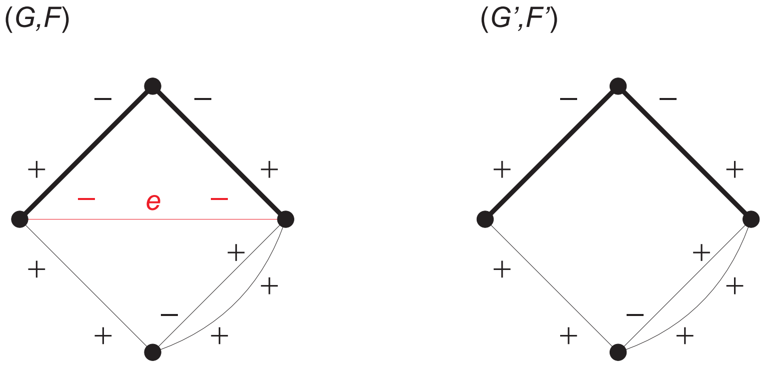
Corresponds to multiplying by -1 the column of $A(G,F)$ corresponding to e.

Delete a node v:



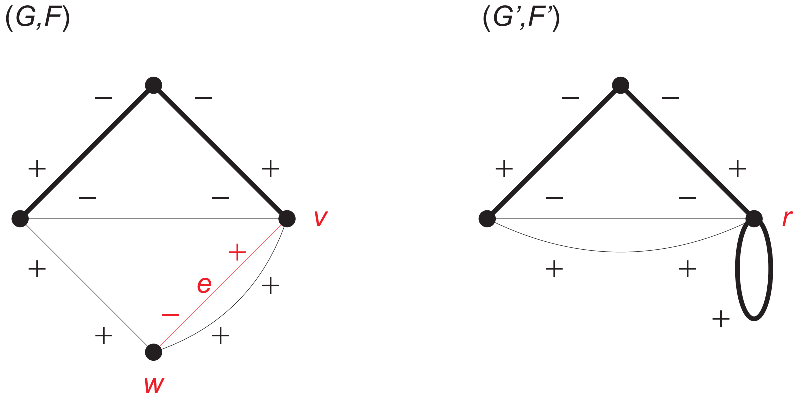
Corresponds to deleting the row of $A(G,F)$ corresponding to v.

Delete an edge e:



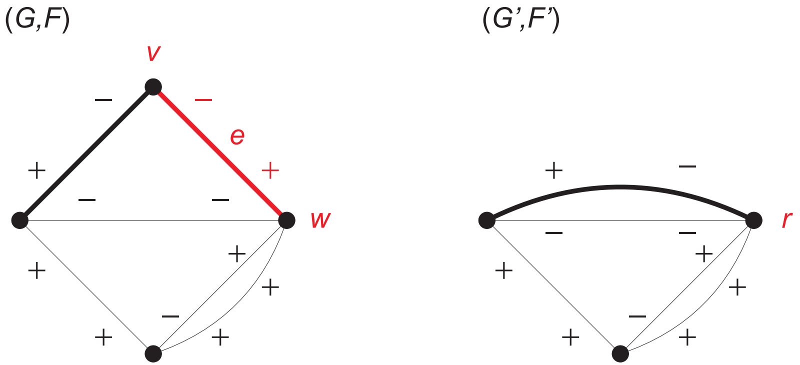
Corresponds to deleting the column of $A(G,F)$ corresponding to e.

Contract a nonloop edge $e = vw \in E \setminus F$:



Corresponds to pivoting the element in position (v, e) of $A(G, F)$, and removing the row corresponding to v and the column corresponding to e.

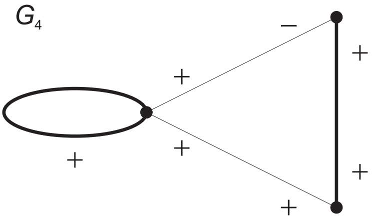
Contract a nonloop edge $e = vw \in F$, where v is incident only with edges in F:



Corresponds to dividing by 2 the row of $A(G,F)$ corresponding to v, pivoting the element in position (v,e), and removing the row corresponding to v and the column corresponding to e.

Definition

Given a bidirected graph $G = (V, E, \sigma)$, and $F \subseteq E$, we call a pair (G', F') a *minor* of (G, F) if it arises from (G, F) by a series of the above operations.



Observation

If (G, F) is the pair G_4, then the matrix $A(G, F)$ contains the matrix A_3 as a minor:

$$\begin{pmatrix} \color{red}{1} & 1 & 2 & 0 \\ -1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} -1 & 1 & 2 & 0 \\ -1 & 1 & 2 & 2 \\ 0 & 1 & 0 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 0 \\ 1 & 2 & 2 \\ 1 & 0 & 2 \end{pmatrix}$$

Thus $A(G_4)$ does not have the Edmonds-Johnson property.

The following is our result.

Theorem

Given a pair (G, F) that satisfies the cycles condition, $A(G, F)$ has the Edmonds-Johnson property if and only if (G, F) does not contain G_4 as a minor.

Definition

Let $\mathscr{C}$ be the family of pairs (G, F) such that:

- (G, F) satisfies the cycles condition,
- (G, F) does not contain G_4 as a minor.

Observation

To prove our theorem we only need to show that the system

$$\begin{aligned} A(G,F)\,x &= c \\ x &\geq 0, \end{aligned}$$

has Chvátal rank at most 1 for every $(G, F) \in \mathscr{C}$ and every integral c.

Observation

Let (G, F) be a pair satisfying the cycles condition. Any irredundant nontrivial Chvátal inequality for

$$\begin{aligned} A(G,F)\,x &= c \\ x &\geq 0, \end{aligned}$$

is equivalent to

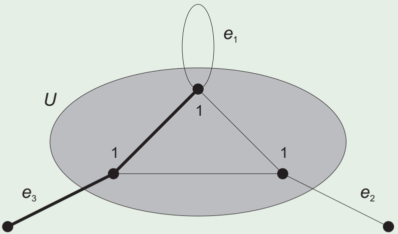
$$x(\delta(U) \setminus F) \geq 1,$$

where:

- $U \subseteq V(G)$ *is connected,*
- $c(U)$ *is odd,*
- *there is no nontrivial partition* U_1, U_2 *of* U *such that all the edges between* U_1 *and* U_2 *are in* F.

We will refer to these inequalities as *odd-cut inequalities*.

Example



Odd-cut:

$$x_{e_1} + x_{e_2} \geq 1.$$

Theorem

If $(G, F) \in \mathscr{C}$, then the polyhedron defined by the system

$$\begin{aligned} A(G,F)\,x &= c \\ x &\geq 0, \end{aligned}$$

and all the odd-cut inequalities is integral.

By the ellipsoid algorithm, for each pair $(G, F) \in \mathscr{C}$, one can minimize in polynomial time any linear function over the integer hull of $b \leq A(G, F)x \leq c$, $l \leq x \leq u$, for all integral vectors b, c, l, u.

Open question: Can we check in polynomial time if a pair (G, F) contains the pair G_4 as a minor?