A branch-and-cut-and-price approach for a two-level location problem

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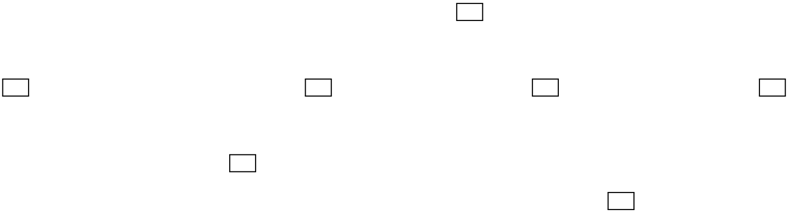
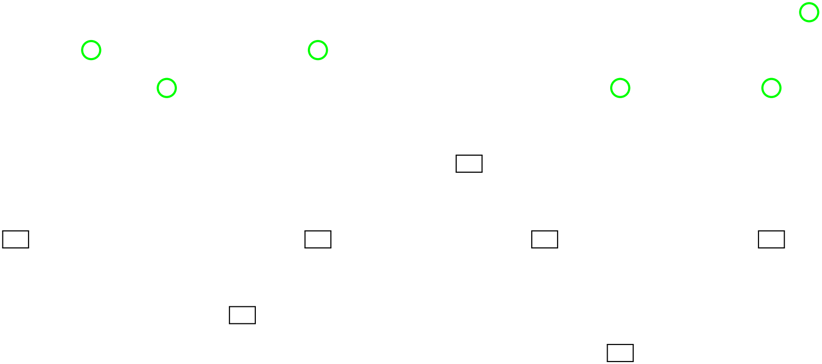
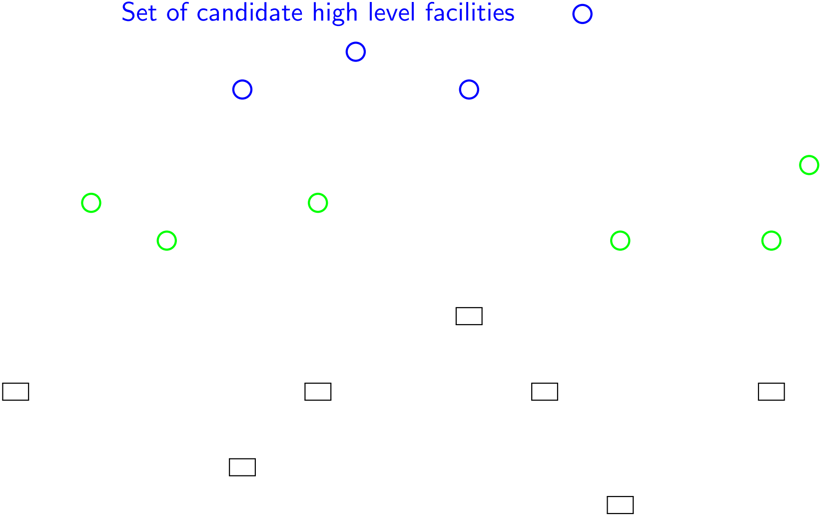


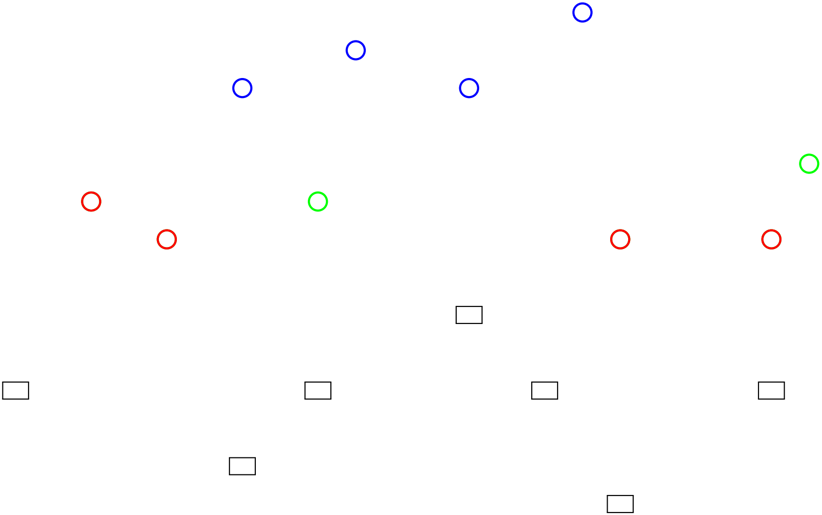
Set of client nodes

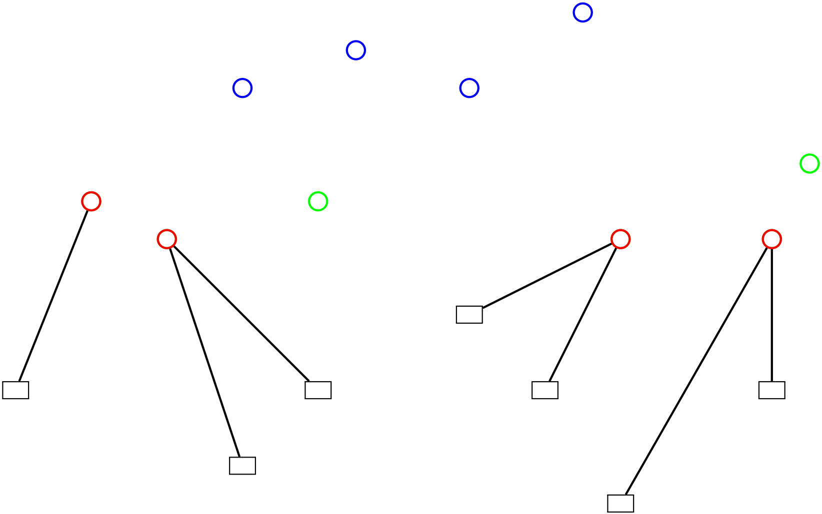
Set of candidate intermediate facilities

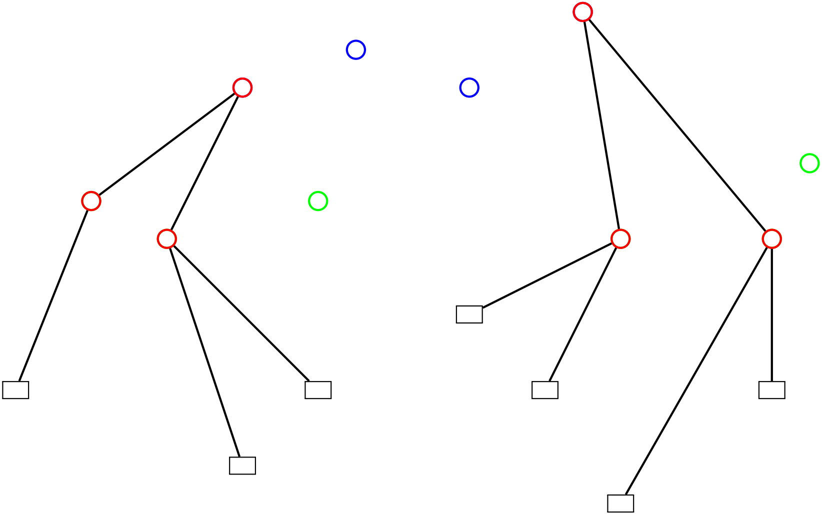


Set of candidate high level facilities









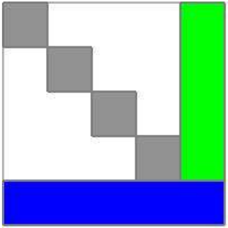
Applications

- In many telecommunication network design problems two different kinds of devices (facilities) must be located
- IP networks (Chamberland, 2007):
  - *access* nodes (clients)
  - *edge* and *core* (facilities)
- Fiber to the home networks (Malucelli and Sircar, 2007)
  - *user houses* (clients)
  - *cabinets* and *central offices* (facilities)
- we consider the problem in which clients are connected to facilities through double stars

Worth studying

It is worth studying because ...

- arises from a practical application
- quite general two-level optimization problem
- particular matrix structure during decomposition ...



Compact model

$$\min \sum_{i\in I, j\in J} d_{ij}x_{ij} + \sum_{j\in J} c_j y_j + \sum_{j\in J, k\in K} l_{jk}w_{jk} + \sum_{k\in K} g_k z_k \tag{1}$$

$$s.t. \sum_{i\in I} a_i x_{ij} \leq b y_j \quad \forall\, j\in J \tag{2}$$

$$\sum_{j\in J} x_{ij} \geq 1 \quad \forall\, i\in I \tag{3}$$

$$y_j \leq \sum_{k\in K} w_{jk} \quad \forall\, j\in J \tag{4}$$

$$\sum_{k\in K} b w_{jk} \leq B z_k \forall\, k\in K \tag{5}$$

and binary variables.

Multifacility case

- Intermediate facilities must be equipped with only one device each;
- for each open facility the device must be chosen among a given set T;
- for each available device a capacity b_t and an installation cost f_t is given, which do not depend on the site.
- economies of scale: $b_t = 2 \cdot b_{t-1}$ and $f_t < 2 \cdot f_{t-1}$

Heuristic idea

- Heuristic approach:
  - solve the problem considering only one device, the one with the highest capacity
  - select, for each facility opened in the optimal solution, the cheapest feasible device to be installed
- The obtained solution is quite far from the optimum: average gap of about 13%, and up to about 51%

$\Longrightarrow$ we need to consider devices in optimizing

Compact model

How the model changes:

- opening variable y_j is replaced with y_{jt}, where $t \in T$, which is equal to 1 if j is open and equipped with device t;
- variable w_{jk} is replaced with w_{jkt}, where $t \in T$, which is equal to 1 if j, equipped with device t, is assigned to k;
- the following constraint is added

$$\sum_{t \in T} y_{jt} \leq 1, \forall j \in J.$$

- For the multifacility case solving the model is more time consuming (871 s. on the average and up to 6600 s. in the worst case)

Compact model

$$\min \sum_{i\in I, j\in J} d_{ij}x_{ij} + \sum_{j\in J} (c_j \quad)y_j + \sum_{j\in J, k\in K} l_{jk}w_{jk} + \sum_{k\in K} g_k z_k$$

$$s.t. \sum_{i\in I} a_i x_{ij} \leq \quad b\ y_j \qquad \forall\ j \in J$$

$$\sum_{j\in J} x_{ij} \geq 1 \qquad \forall\ i \in I$$

$$y_j \leq \sum_{k\in K} w_{jk} \qquad \forall\ j \in J$$

$$\sum_{k\in K} \quad b\ w_{jk} \leq Bz_k \qquad \forall\ k \in K$$

and binary varables.

Compact model

$$\min \sum_{i\in I, j\in J} d_{ij}x_{ij} + \sum_{j\in J, t\in T} (c_j + f_t)y_{jt} + \sum_{j\in J, k\in K, t\in T} l_{jk}w_{jkt} + \sum_{k\in K} g_k z_k$$

$$s.t. \sum_{i\in I} a_i x_{ij} \leq \sum_{t\in T} b_t y_{jt} \qquad \forall\, j \in J$$

$$\sum_{j\in J} x_{ij} \geq 1 \qquad \forall\, i \in I$$

$$y_{jt} \leq \sum_{k\in K} w_{jkt} \qquad \forall\, j \in J\ \forall t \in T$$

$$\sum_{k\in K, t\in T} b_t w_{jkt} \leq Bz_k \qquad \forall\, k \in K$$

$$\sum_{k\in K, t\in T} w_{jkt} \leq 1 \qquad \forall j \in J$$

and binary varables.

Master problem

$$
\begin{aligned}
\min \quad & \sum_{j\in J, t\in T, s\in \mathcal{S}_{jt}} C_s v_s + \sum_{j\in J, k\in K, t\in T} l_{jk} w_{jkt} + \sum_{k\in K} g_k z_k \\
s.t. \quad & \sum_{j\in J, t\in T, s\in \mathcal{S}_{jt}} u_{is} v_s \geq 1 & \forall\ i \in I \quad (\sigma_i) \\
& \sum_{s\in \mathcal{S}_{jt}} v_s \leq \sum_{k\in K} w_{jkt} & \forall\ j \in J, t \in T \quad (\mu_{jt}) \\
& \sum_{j\in J, t\in T} b_t w_{jkt} \leq B\ z_k & \forall\ k \in K \\
& \sum_{k\in K, t\in T} w_{jkt} \leq 1 & \forall\ j \in J
\end{aligned}
$$

Where

- $\mathcal{S}_{jt}$: set of clusters which can be assigned to intermediate facility j equipped with device t;
- $C_s = c_j + f_t + \sum_{i\in s} d_{ij}$: cost of assigning cluster s to facility j, equipped with device t

Pricing problem

- Given a facility j equipped with device t, build a feasible cluster with negative reduced cost.
- Is there a cluster s encoded by binary coefficients u_{is}, such that

$$z = c_j + f_t - \mu_{jt} + \left\{ \max \sum_{i \in I} (d_{ij} - \sigma_i)\, u_{is} : \sum_{i \in I} a_i u_{is} \leq b_t \right\} \leq 0? \quad (6)$$

- A 0–1 knapsack problem has to be solved for each candidate facility j and for each device t.
- Pricing is however to be solved only for each candidate intermediate facility through dynamical programming which provides solution for all the devices.

Discretization

- Weigth $b_t \longrightarrow b_t/b_1$ (number of capacity modules on device t)
- Capacity $B \longrightarrow \lfloor B/b_1 \rfloor$ (number of modules which can be assigned to a high level facility)
- Let $Q = \{q \in \mathbb{Z} : 1 \leq q \leq \lfloor B/b_1 \rfloor\}$
- Substitute variable z_k with variable z_k^q (open facility k and serve q modules)

Master problem

$$
\begin{array}{llr}
\min \displaystyle\sum_{j\in J,t\in T,s\in\mathcal{S}_{jt}} C_s v_s + \sum_{j\in J,k\in K,t\in T} l_{jk} w_{jkt} + \sum_{k\in K} g_k \quad z_k & & \\
\displaystyle\sum_{j\in J,t\in T,s\in\mathcal{S}_{jt}} u_{is} v_s \geq 1 & & \forall\ i \in I \\
\displaystyle\sum_{s\in\mathcal{S}_{jt}} v_s \leq \sum_{k\in K} w_{jkt} & & \forall\ j \in J, t \in T \\
\displaystyle\sum_{j\in J,t\in T} b_t \quad w_{jkt} \leq \quad B \quad\quad z_k & & \forall\ k \in K \\
\displaystyle\sum_{k\in K,t\in T} w_{jkt} \leq 1 & & \forall\ j \in J
\end{array}
$$

Master problem

$$
\begin{aligned}
\min \quad & \sum_{j\in J, t\in T, s\in \mathcal{S}_{jt}} C_s v_s + \sum_{j\in J, k\in K, t\in T} l_{jk} w_{jkt} + \sum_{k\in K} g_k \sum_{q\in Q} z_k^q \\
& \sum_{j\in J, t\in T, s\in \mathcal{S}_{jt}} u_{is} v_s \geq 1 && \forall\ i \in I \\
& \sum_{s\in \mathcal{S}_{jt}} v_s \leq \sum_{k\in K} w_{jkt} && \forall\ j \in J, t \in T \\
& \sum_{j\in J, t\in T} b_t/b_1\ w_{jkt} \leq \lfloor B/b_1 \rfloor \sum_{q\in Q} z_k^q && \forall\ k \in K \\
& \sum_{k\in K, t\in T} w_{jkt} \leq 1 && \forall\ j \in J \\
& \sum_{q\in Q} z_k^q \leq 1 && \forall\ k \in K
\end{aligned}
$$

Valid inequalities

We strenghtened the compact part of the formulation with valid inequalities:
for all $j \in J$ and $k \in K$

$$w_{jkt} \leq \sum_{q \in Q} z_k^q.$$

Moreover,

- Let $N = \lceil \frac{\sum_{i \in I} a_i}{b_1} \rceil$ (min number of capacity modules).
- Then

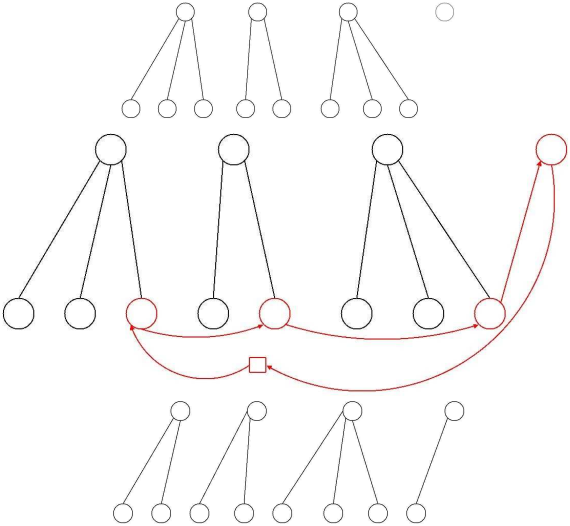
$$\sum_{k \in K} \sum_{q \in Q} q z_k^q \geq N$$

- Hence, for all $p \in Q$,

$$\sum_{k \in K} \sum_{q \in Q} \lceil \frac{q}{p} \rceil z_k^q \geq \lceil \frac{N}{p} \rceil$$

VLSN search

Adapted from Ahuja, Scaparra, Scutellà, Pallottino '05



Branching

During branch-and-bound

- Binary branch on z_k vars
- Binary branch on y_j vars
- Partitioning constraint branch on x_{ij} vars
- Partitioning constraint branch on y_{kt} vars
- Partitioning constraint branch on w_{jkt} vars

Implementation

Implementation:

- C++ implementation
- SCIP as branch-and-cut-and-price framework
- Adaptation of MINKNAP to solve 0-1 KPs (Pisinger '95)
- Automatic cut generation on the root node (compact part)
- Adaptation of the stabilization technique proposed in Uchoa et al (CG2k8)
- Embedded general purpose heuristics + ad-hoc VLSN search
- Ad-hoc 5-level branching

Computational results

Centrino Core2 3GHz processors, tests on 71 instances from facility location problems with up to 200 nodes (Holmberg et al). Compare

- Dual bounds:
  - continuous relaxation
  - lower bound obtained by CPLEX 11 at the root node
  - lower bound produced by Column Generation (CG)
  - lower bound produced by CG, strenghtened by valid inequalities (CRG)
  - lower bound produced by CG, with valid inequalities on the discretized compact part
- Exact solution (time limit 2 hours).
  - CPLEX 11
  - BCP

Results

Dual bounds:

- CPLEX bound between 0.66% and 29.20%, avg. 11.40% in 7.78s
- CRG bound between 0.36% and 27.77%, avg. 10.22% in 25.61s
- CG + discretization bound between 0.00% and 33.82%, avg. 5.99% in 19.44s (avg. improvement 5.71%, worse than CRG in only 4 instances)

Exact solution:

- CPLEX 11 solved 43 instances
- BCP solved 55 instances + 6 below 0.02% + 2 below 0.15%

Conclusions

Conclusions:

- Extended model for first location level + discretized model for second location level works best, providing tight dual bounds
- VLSN search is useful for improving heuristic solutions, providing good primal bounds
- CRG embedded in branch-and-bound is able to solve a large part of the considered instances to proven optimality

To do:

- Solve numerical troubles
- New benchmarks
- Cuts on the extended part of the model