Higher dimensional split closures and lattice point free sets

Kent Andersen. Joint work with Q. Louveaux and R. Weismantel

Institute for Mathematical Optimization,

University of Magdeburg, Germany,

email: andersen@mail.math.uni-magdeburg.de

Outline

◊ Lattice point free convex sets

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- Size measures of split bodies
- Higher dimensional split closures
- Polyhedrality of higher dimensional split closures

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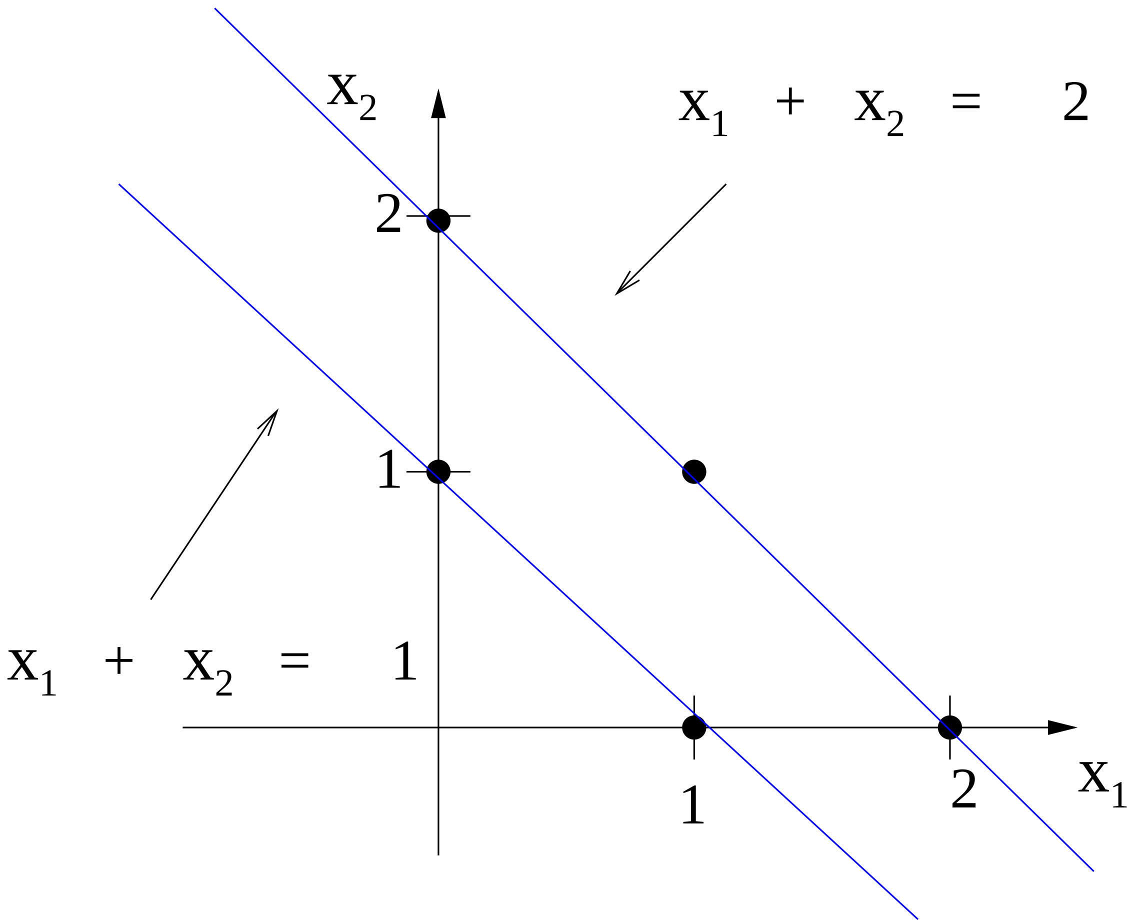
- ◇ Standard split bodies can be used for deriving:
  - ⇒ Mixed integer Gomory cuts (Gomory).
  - ⇒ Mixed integer rounding cuts (Nemhauser and Wolsey).

Split bodies : Examples



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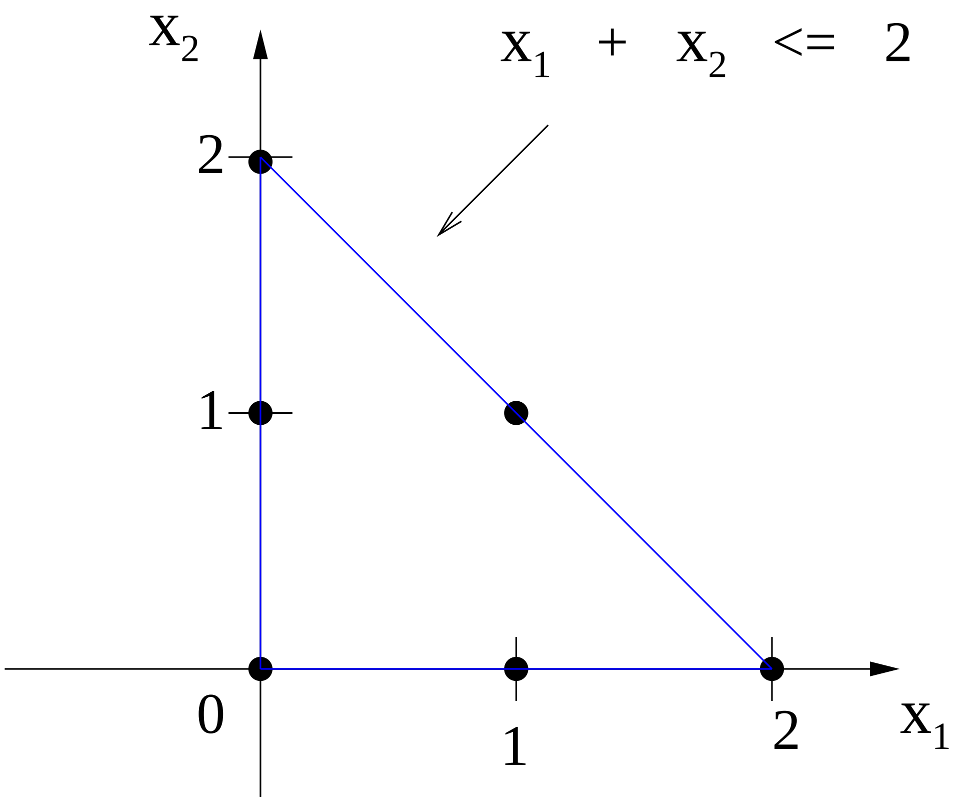

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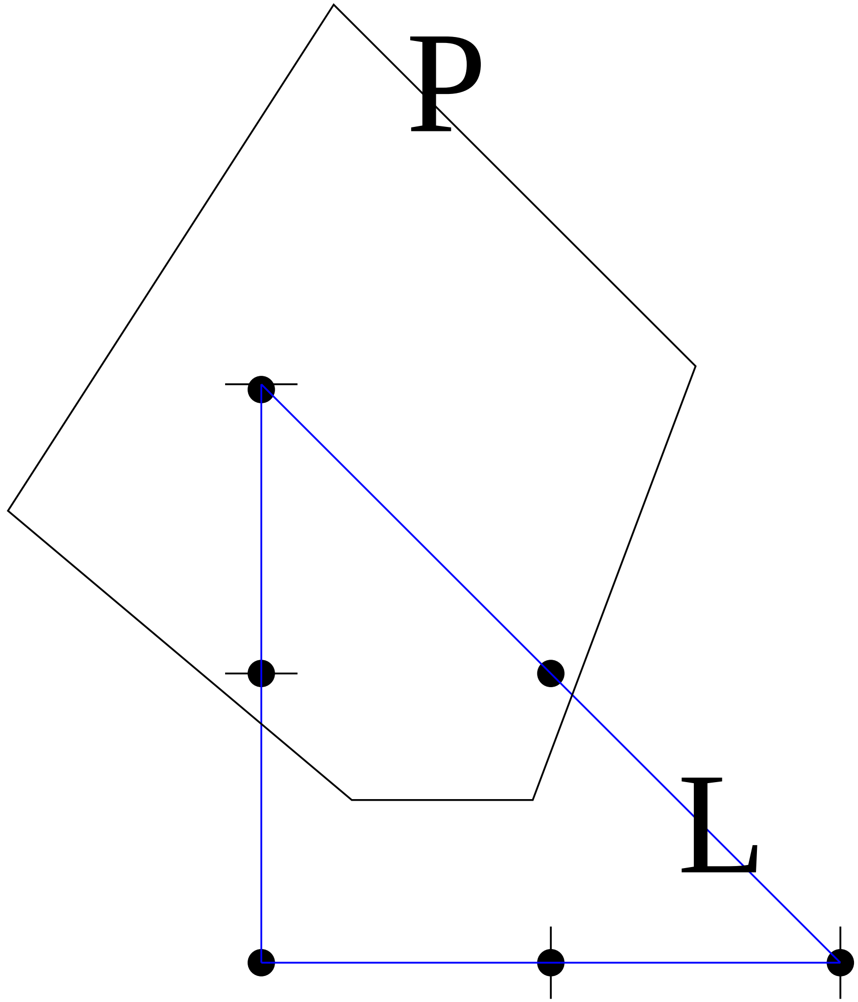
- If L is a standard split set, then valid inequalities for $R(L,P)$ are called split cuts.
- We show (for a general split body L):
  - ⇒ $R(L,P)$ is a rational polyhedron.
  - ⇒ $R(L,P) \neq P$ iff $v \in \mathrm{int}(L)$ for some vertex v of P.
- We call valid inequalities for $R(L,P)$ higher rank split cuts.

A split body and an LP relaxation



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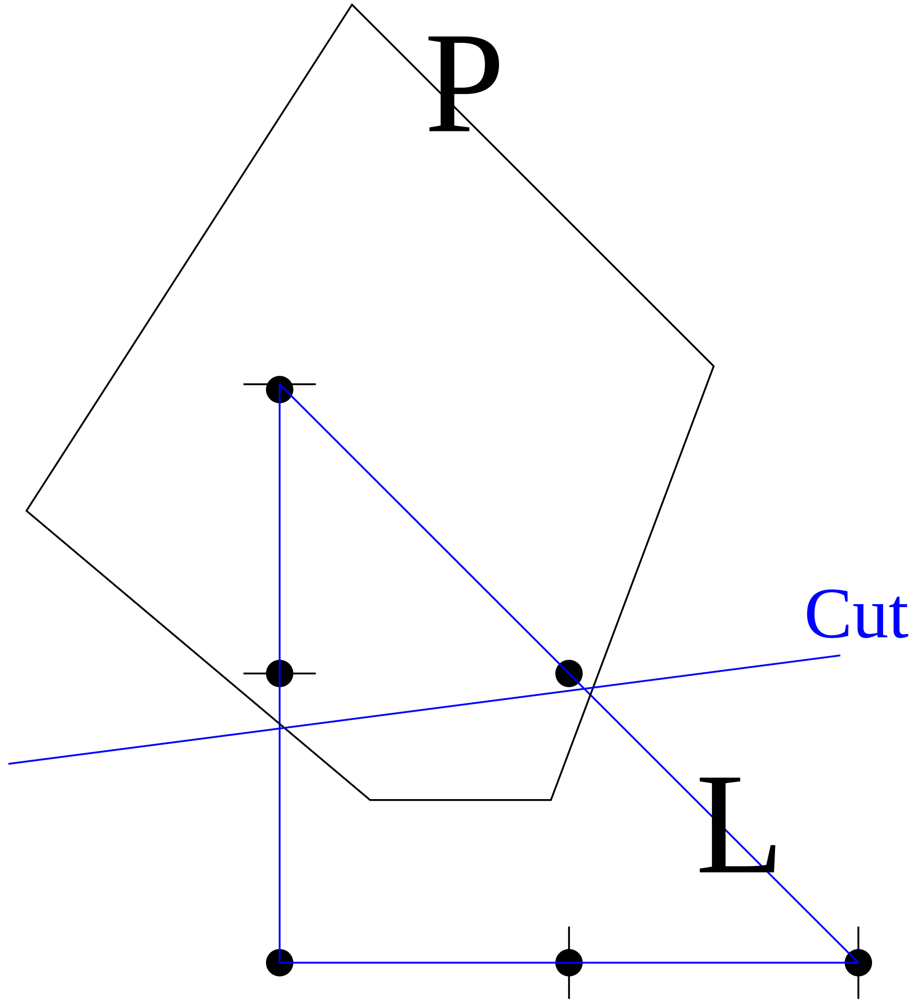

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- We call $\text{ip}_{i,k}$ an intersection point (Balas).

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  onto the space of x-variables.
- Theorem:
  $R(L, Q^i) = \{(x, \lambda) \in Q^i : \sum_{k \in V^{\text{out}}} \frac{\lambda_k}{\beta_{i,k}} \geq 1\}$.

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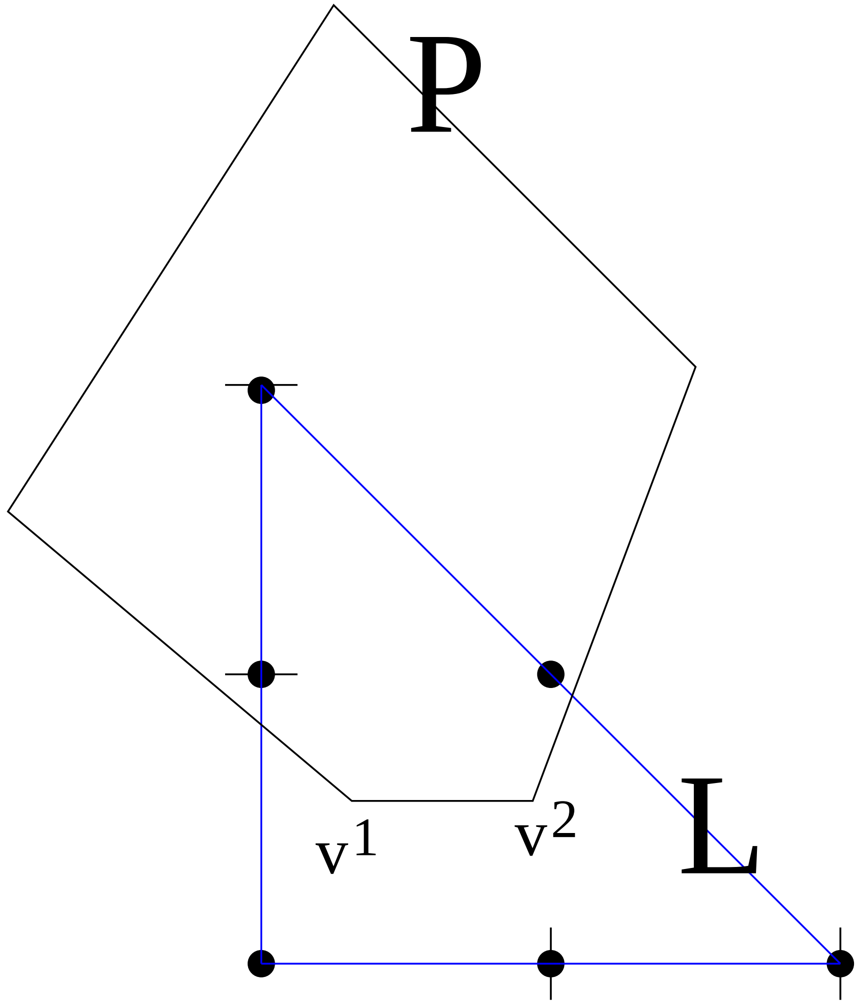
- $R(L, P)$ is completely characterized by the intersection points.

The example continued



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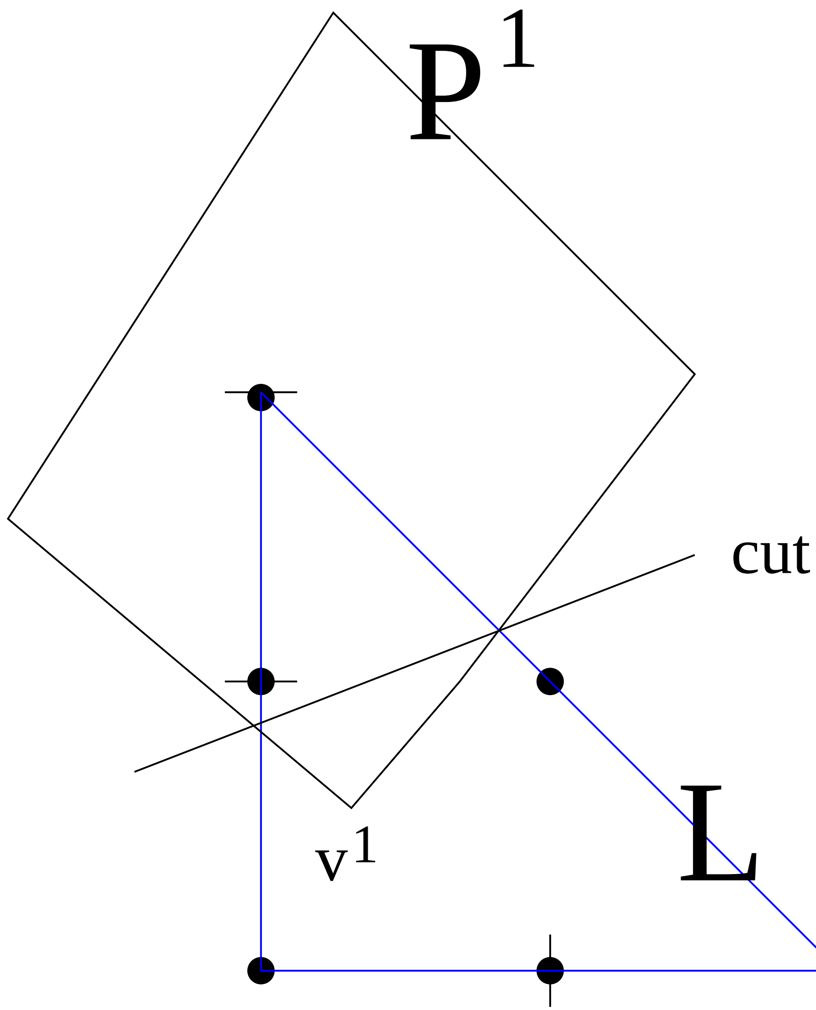

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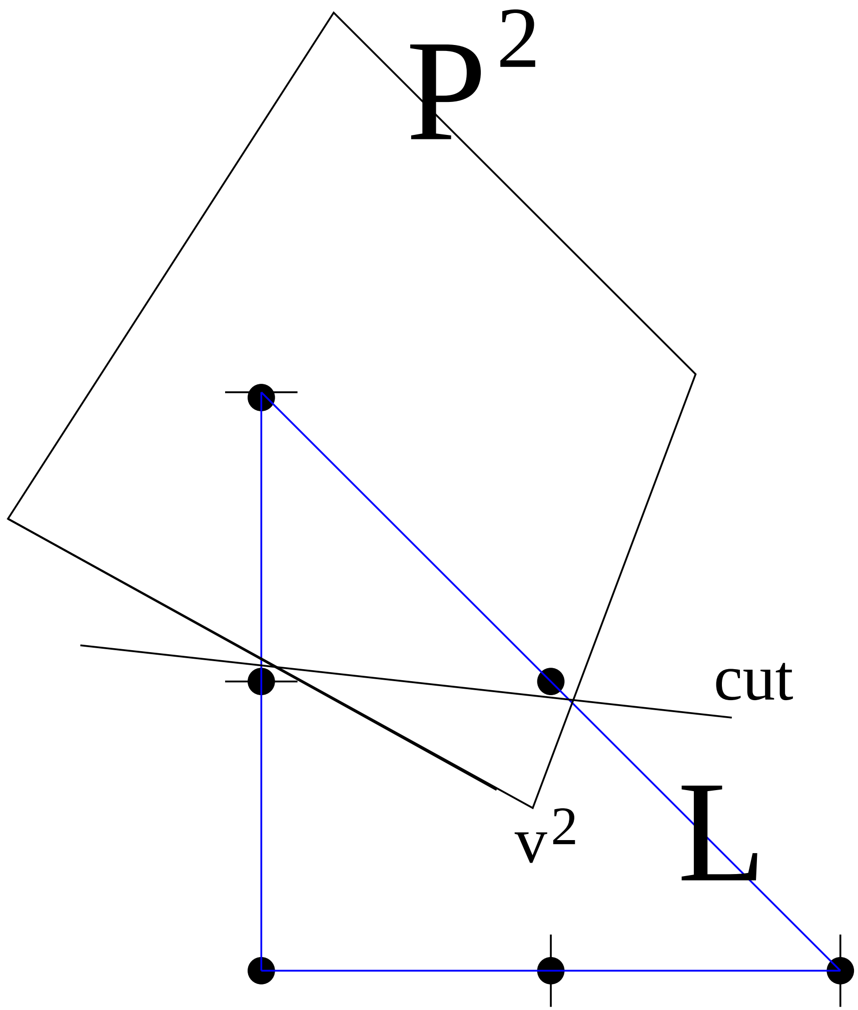

Width measure and width size

- Let $(\pi^k)^T x \geq \pi_0^k$ with $(\pi^k, \pi_0^k) \in \mathbb{Z}^{n+1}$ for $k = 1, 2, \ldots,$ nf be the facets of a split body L.

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- Our example : The set $\{x \in \mathbb{R}^2 : x \geq 0$ and $x_1 + x_2 \leq 2\}$ has max-facet-width equal to two.

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$$\{(x, y) \in \mathbb{Z}^p \times \mathbb{R}_+ : \quad y \leq x_i \text{ for } i = 1, 2, \ldots, p, \quad \textstyle\sum_{i=1}^p x_i + y \leq p \}.$$

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- The valid inequality $y \leq 0$ has width size p.

Higher dimensional split closures

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- For $w = 1$, $\text{Cl}_1(P)$ is known to be a polyhedron.
- Theorem: For any $w \geq 1$ and $S \subseteq \mathcal{L}^w$,

$$\cap_{L \in S} R(L, P) \text{ is a polyhedron.}$$

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- ... define $\beta_{i,k,l} \in]0, 1]$ to be such that the point: $v^i + \beta_{i,k,l}(v^k - v^i)$ satisfies $(\delta^l)^T x \geq \delta_0^l$ with equality.

Proof of Polyhedrality

◊ Theorem: If for all (i,k) and $\beta^* \in]0,1]$, we have:

$$\{\beta_{i,k,l} : l \in I \text{ and } \beta_{i,k,l} \geq \beta^*\} \text{ is finite,}$$

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- The proof is by induction on $|V \setminus V^c| + |E|$.
- We use this theorem on inequalities $\delta^T x \geq \delta_0$ that define facets of $R(L,P)$ for some $L \in S \subseteq \mathcal{L}^w$ to show:

$$\cap_{L \in S} R(L,P) \text{ is a polyhedron.}$$